Homework #1 Solution

Problem 1.

$$B(1, 0.5) \begin{cases} 1 & \text{w.p. } 1/2 \\ 0 & \text{w.p. } 1/2 \end{cases}$$

(5 pts) 1. $$X = 2(B(1, 0.5) - 0.5) \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

$$Y = X^2 = 1 \quad \text{w.p. } 1$$

a. $$\mathrm{Cov}(X, Y) = \underbrace{EX^3}_{0} - \underbrace{EX}_{0}\,\underbrace{EX^2}_{1} = 0$$

$$\Pr(X = i, Y = 1) = \Pr(X = i)$$
$$= \Pr(X = i)\Pr(Y = 1) \qquad \forall\, i = 1, -$$

$\therefore$ X & Y are independent

(5 pts) 2. $$X \begin{cases} 2 & \text{w.p. } 1/4 \\ 1 & '' \\ -1 & '' \\ -2 & '' \end{cases} \qquad Y = X^2 \begin{cases} 4 \\ 1 \end{cases}$$

a. It can be easily seen that $EX = 0$ and $EX^3 = 0$

$$\therefore \mathrm{Cov}(X, Y) = EX^3 - EX\, EX^2 = 0$$

b. $$\Pr(X = 1, Y = 4) = \Pr(\underbrace{X = 1, X^2 = 4}_{\text{This cannot happen.}}) = 0$$

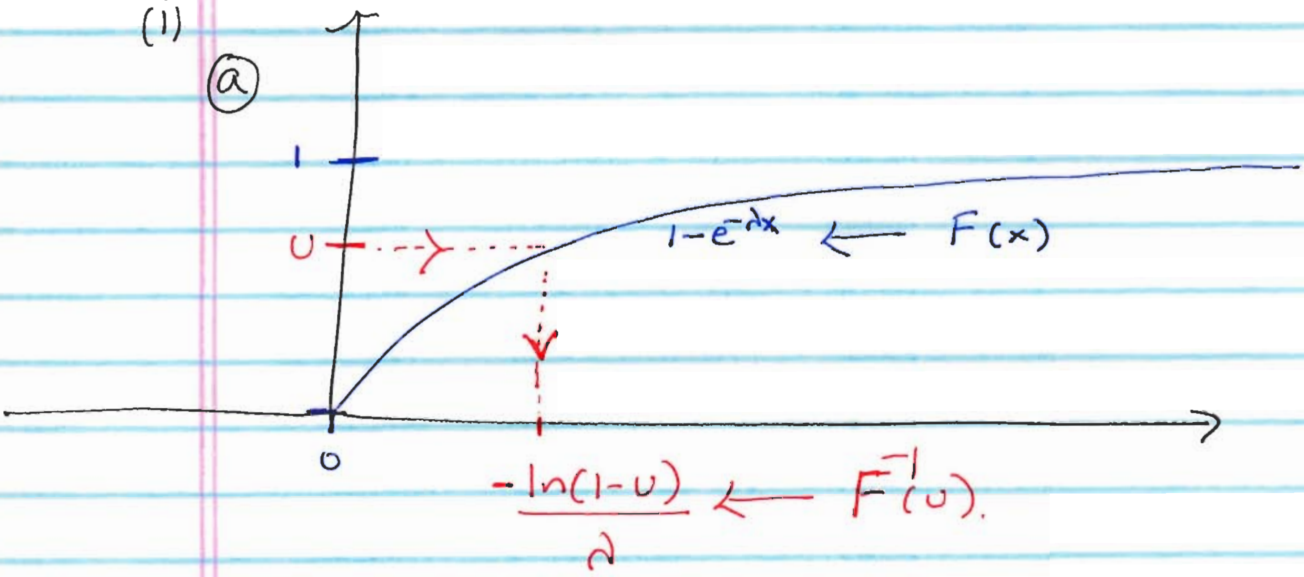
However, $$\Pr(X = 1)\Pr(Y = 4) = \left(\tfrac{1}{4}\right) \cdot \left(\tfrac{1}{2}\right) = \tfrac{1}{8}$$

$\therefore$ X & Y are dependent

(2)

(i)

(a)



$$F(x) = u \iff 1-e^{-\lambda x} = u$$
$$\iff -\lambda x = +\ln(1-u)$$
$$x = \frac{-1}{\lambda}\ln(1-u).$$

(b) $\Pr(F^{-1}(U) \le x)$

$$= \Pr\left(\frac{-1}{\lambda}\ln(1-U) \le x\right)$$
$$= \Pr(\ln(1-U) \geq -\lambda x)$$
$$= \Pr(U \le 1-e^{-\lambda x})$$
$$= 1-e^{-\lambda x}$$

[Distbn. fn. of $U(0,1)$ is x on $[0,1]$]

(ii) $$P_r(X=n) = \begin{cases} q^2 = 1/4 & n=0 \\ 2pq = 1/2 & n=1 \\ p^2 = 1/4 & n=2 \end{cases}$$

$[p = 1/2,\ n=2.]$

To implement (i) part b we need $F^{-1}(\cdot)$ to be defined for "most" of $[0,1]$. The problem with $B(2, 1/2)$ is that

$$F(-\infty, \infty) = \{0, 1/4, 3/4, 1\}$$

or the range of F is a "very small" subset of $[0,1]$. It is true though that F is not one-one, but the ~~problem~~ pertinent problem for simulation is that it is not "almost" onto $[0,1]$.

(iii) Since F is non-decreasing,

$\{z \mid F(z) \geq u\}$ will be of one of the following forms.

(i) (a_u, ∞)

(ii) $[a_u, \infty)$.

If it is (ii), the minimum of the set [i.e. a_u] exists, or in other words the infimum is attained.

We will show that it cannot be of the type (i) as F is right continuous.

Suppose $\{z \mid F(z) \geq u\} = (a_u, \infty)$,

$$F(a_u + 1/n) \geq u \quad \forall n \geq 1$$

$\Rightarrow$ (by right continuity) that

$$F(a_u) \geq u.$$

$$\Rightarrow \quad a_u \in \{z \mid F(z) \geq u\}$$

A Contradiction.

Hence $\{z \mid F(z) \geq u\} \neq (a_u, \infty)$.

$$a_u = \inf \{z \mid F(z) \geq u\}.$$

To prove that

$$\{u \mid F^{-1}(u) \le x\} = (-\infty, F(x)], \quad \forall u \in (0,1)$$

note that the hint shows that the right hand side is a subset of the left handside. Hence we need to show the reverse.

Let $z \in \{u \mid F^{-1}(u) \le x\}$

$$\Rightarrow \quad F^{-1}(z) \le x$$

$$\Rightarrow \inf\{y \mid F(y) \ge z\} \le x$$

$$\Rightarrow \exists\, y_0 \ni \quad y_0 \le x \text{ and } F(y_0) \ge z.$$

$$\Rightarrow \quad F(x) \ge z \quad \text{as} \quad F \text{ is non decreasing.}$$

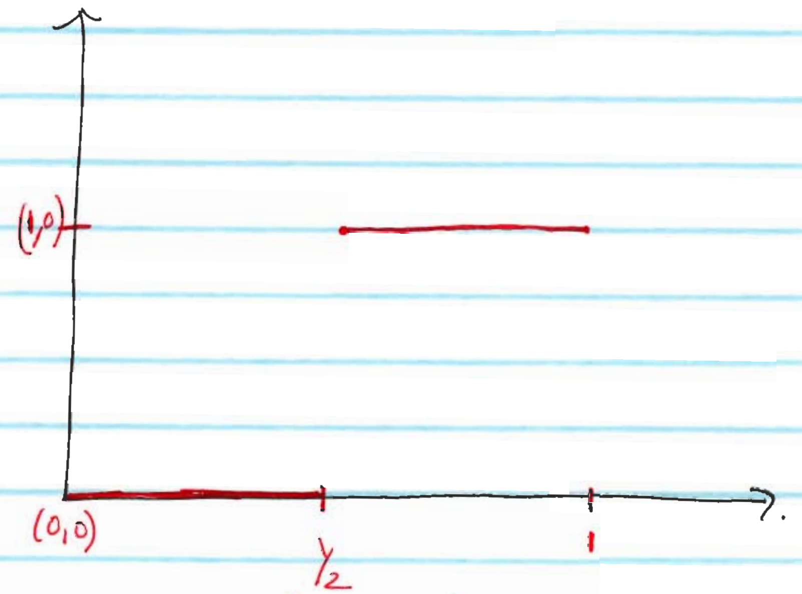
$$\Rightarrow \quad z \in (-\infty, F(z)].$$

(iv) $\inf\{z \mid F(z) \ge u\} = \inf\{z \mid 1 - e^{-\lambda z} \ge u\}$

$$= \inf\left\{z \mid z \ge -\frac{1}{\lambda}\ln(1-u)\right\}$$

$$= \inf\left[-\frac{1}{\lambda}\ln(1-u), \infty\right) = -\frac{1}{\lambda}\ln(1-u). \quad //$$

(v)



(or q for a general Bernoulli)

(vi) $Pr(F^{-1}(U) \le x)$

$= Pr(U \in \{u \mid F^{-1}(u) \le x\})$

$= Pr(U \in (-\infty, F(x)])$ by part(iii)

$= Pr(U \le F(x))$

$= \underline{\underline{F(x)}}$ // Hence $F^{-1}(U) \sim F$.

3 (i)

$$\psi(a) \stackrel{\text{def}}{=} E(X-a)^2$$

$$\psi'(a) = 2E(a-X) = 2(a-EX)$$

$$\psi''(a) = 2 \geq 0$$

Hence equating $\psi'(a) = 0$ we get

$a = EX$, or the minimizer of $\psi(\cdot)$ is EX.

But $E(X-EX)^2 = Var(X)$. Hence

$$Var(X) = \inf_{a \in \mathbb{R}} E(X-a)^2 = \min_{a \in \mathbb{R}} E(X-a)^2$$

(ii) $E(X-a)^2 = E(X-\mu+\mu-a)^2$ where $\mu = EX$.

$$= E(X-\mu)^2 + E(\mu-a)^2 + 2(\mu-a)\underbrace{E(X-\mu)}_{0}$$

$$= E(X-\mu)^2 + (\mu-a)^2$$

$$= \underbrace{Var(X)}_{\substack{\text{independent} \\ \text{of } a.}} + \underbrace{(\mu-a)^2}_{\substack{\text{Always non-ve and } = 0 \\ \text{iff } a = \mu = EX.}}$$

Hence $\min_{a \in \mathbb{R}} E(X-a)^2 = Var(X)$.

(ii)

$Y \downarrow$ \ $X \rightarrow$	0	1
0	$\frac{1}{8}$; $\Pr(X=0 \mid Y=0) = \frac{1}{4}$; $\Pr(Y=0 \mid X=0) = \frac{1/8}{3/8} = \frac{1}{3}$	$\frac{3}{8}$; $\Pr(X=1 \mid Y=0) = \frac{3}{4}$; $\frac{3}{5}$
1	$\frac{1}{4}$; $\frac{1}{2}$; $\Pr(Y=1 \mid X=0) = \frac{1/4}{3/8} = \frac{2}{3}$	$\frac{1}{4}$; $\frac{1}{2}$; $\frac{2}{5}$

$\left.\Pr(Y=0) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}\right\}$ $Ber(\frac{1}{2})$

$\left.\begin{array}{l} E(X \mid Y=0) = 3/4 \\ Var(X \mid Y=0) = 3/16 \end{array}\right\} Ber(3/4)$

$\left.\Pr(Y=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\right\}$

$\left.\begin{array}{l} E(X \mid Y=1) = 1/2 \\ Var(X \mid Y=1) = 1/4 \end{array}\right\} Ber(1/2)$

$Ber(5/8) \left\{ \begin{array}{ll} \Pr(X=0) = \frac{1}{8} + \frac{1}{4} = 3/8 & \Pr(X=1) = \frac{3}{8} + \frac{1}{4} = 5/8 \end{array}\right.$

$Ber(2/3) \left\{ \begin{array}{l} E(Y \mid X=0) = 2/3 \\ Var(Y \mid X=0) = 2/9 \end{array}\right.$

$\left.\begin{array}{l} E(Y \mid X=1) = 2/5 \\ Var(Y \mid X=1) = 6/25 \end{array}\right\} Ber(2/5)$

(a) $E(X) = 5/8$ $\qquad E(E(X \mid Y)) = \frac{3}{4} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = 5/8.$

$E(Y) = \frac{1}{2}$ $\qquad E(E(Y \mid X)) = \frac{2}{3} * \frac{3}{8} + \frac{2}{5} * \frac{5}{8} = \frac{1}{2}.$

(b) $\mathrm{Cov}\left(X - E(X|Y),\ E(X|Y)\right)$

$$= E\left(\left(X - E(X|Y)\right)E(X|Y)\right)$$

as $E\left(X - E(X|Y)\right) = 0$ part a.

$Y \backslash X$	0	1
0	$-3/4$, $3/4$	$1/4$, $3/4$
1	$-1/2$, $1/2$	$1/2$, $1/2$

$X - E(X|Y)$ → red values

$E(X|Y)$ → blue values.

Hence $E\left[\left(X - E(X|Y)\right)E(X|Y)\right]$

$$= \cancel{-\frac{9}{16} * \frac{1}{8}} + \cancel{\frac{3}{16} * \frac{3}{8}} \cancel{- \frac{1}{4} * \frac{1}{4}} + \cancel{\frac{1}{4} * \frac{1}{4}}$$

$= 0.$

(c) As $X \sim Ber(5/8)$

$Var(X) = 15/64$

$E\,Var(X|Y) = \frac{3}{16} * \frac{1}{2} + 1/4 * \frac{1}{2} = 7/32$

$Var(E(X|Y)) = \left(\frac{3}{4} - \frac{1}{2}\right)^2 * \frac{1}{2} * \frac{1}{2}$

$= \frac{1}{16} * \frac{1}{4} = \frac{1}{64}$

$E\,Var(X|Y) + Var(E(X|Y)) = 7/32 + 1/64 = 15/64 = Var(X)$ //

As $Y \sim Ber(1/2)$, $Var(Y) = 1/4$.

$E\,Var(Y|X) = \frac{2}{9} * 3/8 + \frac{6}{25} * \frac{5}{8}$

$= \frac{1}{12} + \frac{3}{20} = \frac{14}{60} = 7/30$ //

$Var(E(Y|X)) = \left(\frac{2}{3} - \frac{2}{5}\right)^2 * 3/8 * 5/8$

$= \frac{4^2}{(15)^2} * \frac{15}{8^2} = 1/60$

$E\,Var(Y|X) + Var(E(Y|X)) = 1/60 + 7/30 = 15/60 = 1/4$ //

Problem 4 (10 pts, 2.5 pts each)

A. When $3 \le x \le 4$, $F(x) = \frac{2\cdot 3\cdot x - 3\cdot 4}{24}$

$\therefore F(4) = \frac{12}{24} = \frac{1}{2}$ but $F(4+) = 1$

$\therefore F(x)$ is not right continuous

Hence Not a distribution function.

B. i

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \le x < 2 \\ 1/2 & 2 \le x < 3 \\ 1 & \end{cases}$$

F Satisfies i) $0 \le F(x) \le 1$ $\forall x$

ii) Nondecreasing

iii) Right Continuous

iv) $F(-\infty) = 0$ $F(+\infty) = 1$

Hence a distribution function

ii a discrete distribution function. $\because \sum \left[F(x) - F(x-)\right] = 1$

iv Consider $X = \begin{cases} 1 & \text{w.p } 1/6 \\ 2 & \text{w.p. } 1/3 \\ 3 & \text{w.p. } 1/2 \end{cases}$

Its distribution is exactly same with F.

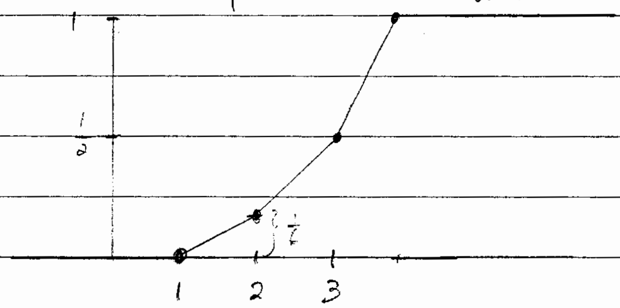
$\therefore EX = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{2} = \frac{1+4+9}{6} = \frac{7}{3}$

$EX^2 = 1\cdot\frac{1}{6} + 2^2\cdot\frac{1}{3} + 3^2\cdot\frac{1}{2} = \frac{1+8+27}{6} = 6$

$Var(X) = 6 - \frac{49}{9} = \frac{5}{9}$

Or $EX = \int_{-\infty}^{\infty} x\, dF(x) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{2} = \frac{7}{3}$

C. $$F(x) = \begin{cases} 0 & x < 1 \\ (x-1)/6 & 1 \le x < 2 \\ (2x-3)/6 & 2 \le x < 3 \\ (x-2)/2 & 3 \le x < 4 \\ 1 & \text{o.w.} \end{cases}$$



i. Satisfy the required properties as in B.

ii. Continuous ∴ no jump $\Leftrightarrow \sum \left[F(x) - F(x-) \right] = 0$

iv. $EX = \int_{-\infty}^{\infty} x \, dF(x)$ where X is associated w/ F.

$$= \int_1^{2-} x \cdot \frac{1}{6} \, dx + \int_2^{3-} x \cdot \frac{1}{3} \, dx + \int_3^{4-} x \cdot \frac{1}{2} \, dx$$

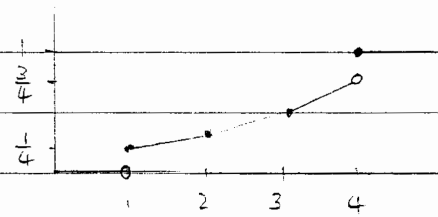
from pdf

$$= \frac{1}{12}(2^2 - 1) + \frac{1}{6}(3^2 - 2^2) + \frac{1}{4}(4^2 - 3^2) = \frac{34}{12} = \frac{17}{6}$$

Likewise $EX^2 = \frac{26}{3}$

∴ $Var(X) = 0.6389$

D. $F(x) = \frac{1}{2}$ (Distribution in C) $+ \frac{1}{4}$ when $1 \le x < 4$



i Satisfy the properties as in B.

ii mixture ∴ 2 jumps at $x = 1, 4$ i.e. $0 < \Sigma[F(x) - F(x^-)] < 1$

iii $F = \frac{1}{2} F_C(x) + \frac{1}{2} F_D(x)$

$F_C(x)$ as in C

$$F_D(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

iv. Consider a r.v $X_D = \begin{cases} 1 & \text{w.p. } 1/2 \\ 4 & \text{w.p } 1/2 \end{cases}$

which corresponds to $F_D(x)$.

Then $EX_D = \frac{5}{2}$ $Var(X_D) = \frac{9}{4}$

Let X be a r.v associated w/ the distribution

Then

$$EX = 1/2 \cdot (\text{Expectation in } C) + 1/2 \cdot EX_D$$

$$= 1/2 \cdot (17/6) + 1/2 \cdot (5/2)$$

$$= 8/3$$

$$Var(X) = 1/2 (\text{Variance in } C) + 1/2\, Var(X_D)$$

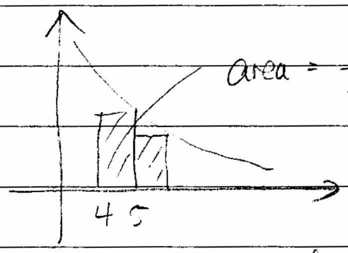
$$+ 1/2 \cdot 1/2 \left(\text{Expectation in } C - EX_D\right)^2$$

$$= 53/36$$

Problem * 1 (extra 5pts)

$$P(X=n) = \begin{cases} \dfrac{k}{n^2 \log n \cdot (\log(\log n))^2} & n = 5, 6, \dots \text{ for some } k \\ 0 & \text{o.w.} \end{cases}$$

a. Consider $f(x) = \dfrac{k}{x \log x (\log(\log x))^2}$, a decreasing fn of x



area $= \dfrac{k}{5 \log 5 (\log(\log 5))^2}$

$$EX = \sum_{n=5}^{\infty} n \cdot \frac{k}{n^2 \log n (\log(\log n))^2}$$

$$= \frac{k}{5 \log 5 (\log(\log 5))^2} + \cdots < \int_4^{\infty} \frac{k}{x \log x (\log(\log x))^2} dx$$

By change of variable $\log(\log x) = y$

$$\frac{1}{\log x} \cdot \frac{1}{x} dx = dy$$

Thus Right Integral

$$= \int_{\log(\log 4)}^{\infty} \frac{k}{y^2} dy$$

$$= \left[-k y^{-1} \right]_{\log(\log 4)}^{\infty} = \text{constant} < \infty.$$

b. $E(X^{1+\varepsilon}) = \infty \quad \forall \varepsilon > 0$

For sufficiently large $N(\varepsilon) \in \mathbb{N}$, we can show that if $n \geq N(\varepsilon)$,

$$\log(\log n) \leq \log n < n^{\varepsilon/3} \qquad \text{--- } (*)$$

First inequality follows from $\log x < x$ if $x > 0$ & $\log(\cdot)$ is an increasing function

For sencond inequality, consider $n < e^{n^{\varepsilon/3}}$ and Taylor expansion of the right hand side.

From $(*)$, $n^{\varepsilon} = n^{\varepsilon/3} \cdot \left(n^{\varepsilon/3}\right)^2 > \log n \left(\log(\log n)\right)^2$ if $n \geq N(\varepsilon)$

Hence,

$$E(X^{1+\varepsilon}) \geq \sum_{n=N(\varepsilon)}^{\infty} n^{1+\varepsilon} \frac{k}{n^2 \log(n)\left(\log(\log n)\right)^2} > \sum_{n=N(\varepsilon)}^{\infty} \frac{k}{n} = \infty$$