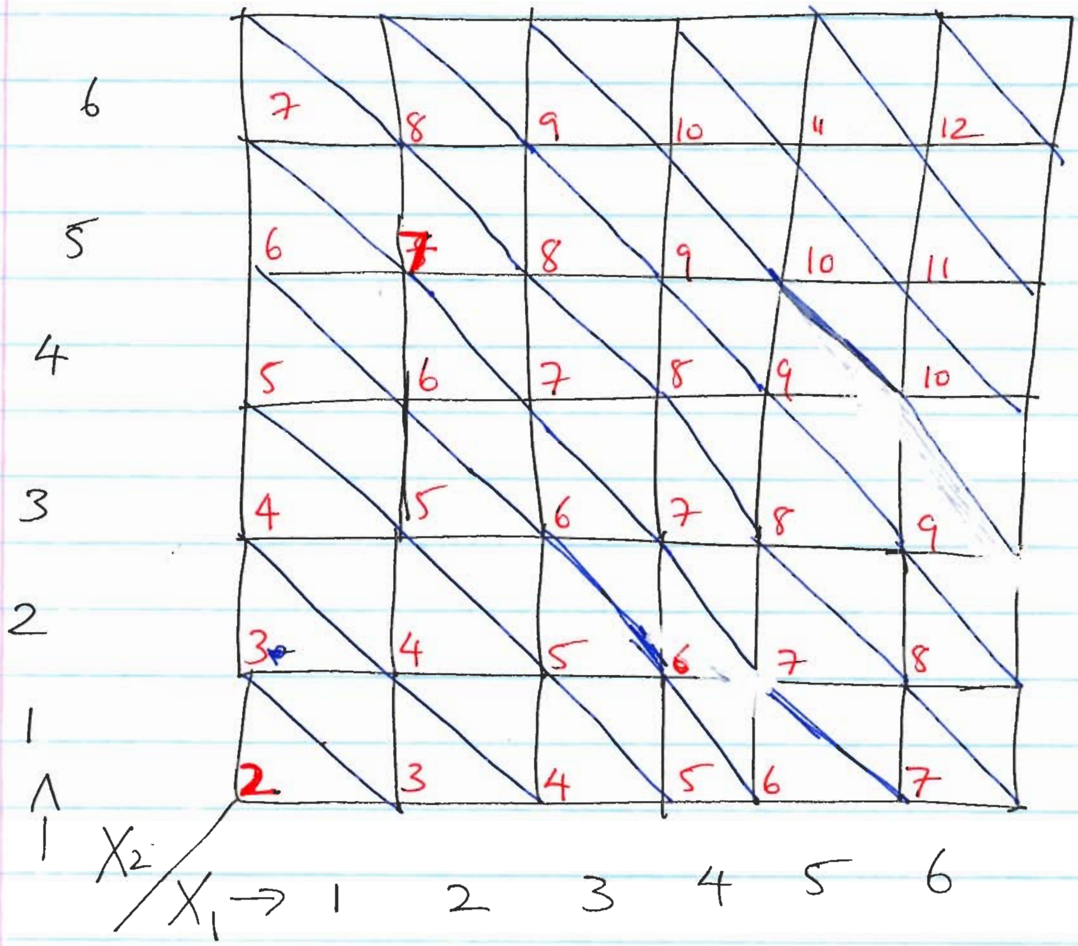
Problem 1

1.



36 Squares equal probability of $1/36$.

Values in Red are the values of $X_1 + X_2$.

Note: Values of $X_1 + X_2$ assigned to boxes on the same diagonals [Blue Lines] are the same. Hence

$$Pr(X_1 + X_2 = n) = \frac{6 - |7 - n|}{36}, \quad n = 2, \dots 12$$

Note that as X_1 & X_2 are symmetrical about 3.5, $X_1 + X_2$ is about 7.

2. Method 1

Poisson with parameter λ has a m.g.f. of

$$\phi(t) = e^{\lambda(e^t - 1)}$$

Hence $X_1 + X_2 + \cdots X_n$ has a m.g.f. of

$$(\phi(t))^n = \left[e^{\lambda(e^t-1)}\right]^n = e^{n\lambda(e^t-1)}$$

which is nothing but the m.g.f of Poisson($n\lambda$)

Method 2.

We shall first prove that if $X \sim$ Poisson(α) and $Y \sim$ Poisson (λ), then $X+Y \sim$ Poisson($\alpha+\lambda$) if X is independent of Y.

$$P_r(X+Y=n) = \sum_{k=0}^{n} P(X=k)\,P(Y=n-k)$$

$$= \sum_{k=0}^{n} \frac{e^{-\alpha}\alpha^k}{k!} \frac{e^{-\lambda}\lambda^{n-k}}{(n-k)!}$$

$$= e^{-(\alpha+\lambda)} \frac{(\alpha+\lambda)^n}{n!} \sum_{k=0}^{n} \left(\frac{\alpha}{\alpha+\lambda}\right)^k \left(\frac{\lambda}{\alpha+\lambda}\right)^{n-k} \frac{n!}{k!\,(n-k)!}$$

$$= \frac{e^{-(\alpha+\lambda)}(\alpha+\lambda)^n}{n!} \left(\frac{\alpha}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda}\right)^n$$

$$= \frac{e^{-(\alpha+\lambda)}(\alpha+\lambda)^n}{n!} \quad \forall\ n \geq 0.$$

Hence $X + Y \sim$ Poisson $(\alpha+\lambda)$.

Now by induction on n, one can show that

$$X_1 + \dots + X_n = S_n \sim \text{Poisson}(n\lambda) //.$$

The Poisson Process Connection

If we have inter arrival times being iid Expon with mean $1/\lambda$, then the no. of arrivals in ~~time of~~ $[0,1]$, $[1,2]$ ... $[n-1,n]$ are iid Poisson (λ) and the no. of arrivals in $[0,n]$ is Poisson $(n\lambda)$.

3. <u>Method 1</u>

M.g.f. of $N(\mu, \sigma^2)$ is $e^{\mu t + \frac{\sigma^2 t^2}{2}}$

Hence if $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ then $X_1 + \dots X_n$

has a m.g.f of

$$\left[e^{\mu t + \frac{\sigma^2 t^2}{2}}\right]^n$$

$$= e^{n\mu t + \frac{n\sigma^2 t^2}{2}}$$

which is the m.g.f. of $N(n\mu, n\sigma^2)$

<u>Method 2</u>

We shall show first that if X, Y are iid $N(0,1)$ then

$X+Y \sim N(0,2)$.

This will imply that if $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\lambda, \sigma^2)$ then $X+Y \sim N(\mu+\lambda, 2\sigma^2)$ if X and Y are independent.

This is so as $\left(\frac{X-\mu}{\sigma}\right)$ and $\left(\frac{Y-\alpha}{\sigma}\right)$ will be iid $N(0,1)$. Hence $\frac{X-\mu}{\sigma}+\frac{Y-\alpha}{\sigma}$

$$= \frac{X+Y-(\mu+\alpha)}{\sigma} \sim N(0,2)$$

or $X+Y \sim N(\mu+\alpha, 2\sigma^2)$.

By induction one can show that $X_i \overset{iid}{\sim} N(\mu,\sigma^2)$

$\Rightarrow \sum_1^n X_i \sim N(n\mu, n\sigma^2)$.

Now we prove $\longrightarrow$ that $X,Y \overset{iid}{\sim} N(0,1) \Rightarrow X+Y \sim N(0,2)$

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} \phi(s)\,\phi(t-s)\,ds$$

$$= \int_{-\infty}^{\infty} \frac{e^{-s^2/2}}{\sqrt{2\pi}} \frac{e^{-\frac{(t-s)^2}{2}}}{\sqrt{2\pi}}\,ds$$

$$= \int_{-\infty}^{\infty} \frac{e^{-[s^2-st+t^2/2]}}{(\sqrt{2\pi})^2}\,ds = \frac{e^{-t^2/4}}{\sqrt{2\pi}\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-[s^2-st+t^2/4]}}{\sqrt{2\pi}\,(1/\sqrt{2})}\,ds$$

$$= \frac{e^{-t^2/4}}{\sqrt{2\pi}\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-(s-t/2)^2}}{\sqrt{2\pi}\,(1/\sqrt{2})}\,ds$$

$$= \frac{e^{-t^2/4}}{\sqrt{2\pi}\,\sqrt{2}}$$ as the integral is 1 [because the integrand is the density of $N(t/2, 1/2)$]

Hence $X+Y \sim N(0,2)$.

4. It can be shown that the c.fn. [characteristic fn.] of a ~~fo~~ Cauchy $(0,1)$ is

$$e^{-|t|} \quad \forall t.$$

Hence the c.fn. of $\frac{\sum_1^n X_i}{n}$, $X_i \overset{iid}{\sim}$ Cauchy$(0,1)$

is given by $\left[e^{-\frac{|t|}{n}}\right]^n = e^{-|t|}$.

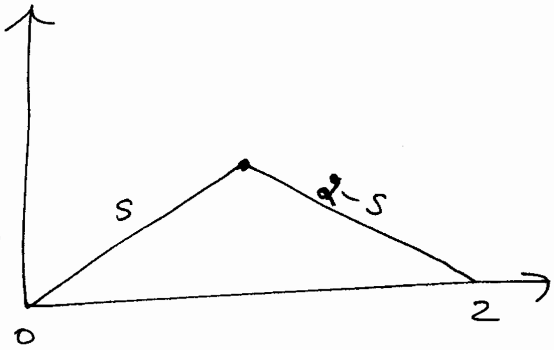
Hence $\frac{\sum_1^n X_i}{n} \sim$ Cauchy $(0,1)$.

This in particular implies that if the variance existed for a Cauchy $(0,1)$, then the Variance of the Sample Mean will be the same as X_1.

But that contradicts the fact that $Var\left(\frac{\sum_1^n X_i}{n}\right) = \frac{Var(X_1)}{n}$. Hence Cauchy $(0,1)$ does not have a finite variance.

2.8 We know that if $X_i \overset{iid}{\sim} U(0,1)$ then $X_1 + X_2$ has a density which is a triangle

$$f^{(2)}(s) = \begin{cases} s & 0 \le s \le 1 \\ 2-s & 1 \le s \le 2. \end{cases}$$



See the note on convolution for a proof.

We use this to get the density $f^{(3)}(\cdot)$ of $X_1 + X_2 + X_3$. Moreover, as X_i's are symmetrical about 0.5 $X_1 + X_2 + X_3$ are symmetrical about 1.5. Hence

$$f^{(3)}(s) = f^{(3)}(3-s) \quad \forall \; s$$

We shall use the above to simplify our work.

Note, $f^{(3)}(s) = \int_0^1 f_{X_3}(t)\, f^{(2)}(s-t)\, dt$

For $0 \le s \le 1$, $0 \le s-t \le 2$ if and only if $0 \le t \le s$.

Hence $f^{(3)}(s) = \int_0^s 1 \cdot (s-t)\, dt = \dfrac{s^2}{2}$

For $1 \le s \le 2$

$$f^{(3)}(s) = \int_0^1 f^{(2)}(s-t)\,dt$$

But if $t < s-1$ $s-t > 1$.

Hence

$$f^{(3)}(s) = \int_0^{s-1} (2-(s-t))\,dt + \int_{s-1}^{1} (s-t)\,dt$$

$$= \frac{1}{2} - \frac{(2-s)^2}{2} + \frac{1}{2} - \frac{(s-1)^2}{2}$$

$$= 1 - \frac{(2-s)^2}{2} - \frac{(s-1)^2}{2}$$

$$= 3/4 - (s-3/2)^2$$ [Which is symmetrical about 1.5].

Hence combining

$$f^{(3)}(s) = \begin{cases} s^2/2 & 0 \le s \le 1 \\ 3/4 - (s-3/2)^2 & 1 \le s \le 2 \\ \dfrac{(3-s)^2}{2} & 2 \le s \le 3 \end{cases}$$

by symmetry.

$$F^{(3)}(s) = \int_0^s t^2/2 \, dt = t^3/6 \qquad 0 \le s \le 1$$

$$F^{(3)}(s) = F^{3}(1) + \int_1^s \left[3/4 - (t - 3/2)^2 \right] dt$$

$$= 1/6 + 3/4\,(s-1) - \frac{(s - 3/2)^3}{3} - \frac{1}{24}$$

$$= \frac{3}{4}s - \frac{(s-1.5)^3}{3} - \frac{15}{24} = \underline{\underline{\frac{3}{4}(s - 3/2) - \frac{(s-1.5)^3}{3} + \frac{1}{2}}}.$$

$$= \frac{s^3 - 3(s-1)^3}{6} \qquad \forall \quad 1 \le s \le 2.$$

Why is this form interesting ??

$$F^{(3)}(s) = F^{(3)}(2) + \int_2^s \frac{(3-t)^2}{2} dt$$

$$= \frac{5}{6} + \frac{1}{6} - \frac{(3-s)^3}{6} = 1 - \frac{(3-s)^3}{6}$$ [Can you see the symmetry ?]

which can be complicated to the form in Bowers.

(b) $E(S) = 1.5$ is obvious as S is symmetrical about 1.5. Also, see

$$\boxed{S \overset{d}{=} 3 - S \Rightarrow ES = 3 - ES \Rightarrow ES = 1.5.}$$

Benefit Amount (b_k)	Number (n_k) Covered
1	8000
2	3500
3	2500
5	1500
10	500

$q = 0.02$

Reinsurance Rate = 125% of q

Problem: Minimize Pr(Retained Claims + Reinsurance Premium > 825).

Step 1 The above probability

$$= \Pr(S + RP > 825)$$

where

$S \rightarrow$ Retained Claims

$RP \rightarrow$ Reinsurance Premium.

$TSA \rightarrow$ Total Sum Assured [in this case 35,000].

$$\boxed{RP = 1.25 * [TSA * q - ES]}$$

as Reinsurance rate is 125% of q. Verify!

$$= \Pr(S + 1.25[TSA * q - ES] > 825)$$

$$= \Pr(S - ES > 825 - [TSA * q - ES] * 25\% - TSA * q).$$

$$\approx 1 - \Phi\left(\frac{825 - [TSA * q - ES] * 25\% - TSAq}{\sqrt{Var(S)}}\right)$$

As Φ is monotone, the minimization problem is approximately [due to the CLT] the same as maximizing

$$\frac{825 - TSA * q * 1.25 + ES * 25\%}{\sqrt{Var(S)}}$$

Fixed Quantities

Variable Quantities w.r.t. Rentention limit.

Let d bethe retenention limit.

Then $E(S) = q * \sum_{k=1}^{5} (b_k \wedge d) * n_k$

and $Var(S) = q*(1-q) \sum_{K=1}^{5} (b_k \wedge d)^2 * n_k$

It is easy to see that

$$E(S) = \begin{cases} q * d * \sum_1^5 n_K & d \leq 1 = b_1 \\ q*[b_1 * n_1 + d * \sum_2^5 n_k] & b_1 < d \leq b_2 \\ \vdots & \\ q*[\sum_1^4 b_k n_k + d * n_5] & b_4 < d \leq b_5 \\ q * \sum_1^5 b_K n_K & b_5 < d \end{cases}$$

or more importantly just notice that

ES is linear in d when d is between two consecutive b_k's [b_k's are assumed to be ordered].

Also

$$\frac{Var(S)}{q(1-q)} = \begin{cases} d^2 \sum_{k=1}^{5} n_k & d \le 1 = b_1 \\ [b_1^2 * n_1 + d^2 \sum_{2}^{5} n_k] & b_1 < d_1 \le b_2 \\ \vdots & \vdots \\ [\sum_{1}^{4} b_k^2 n_k + d^2 n_5] & b_4 < d \le b_5 \\ \sum_{1}^{5} b_k^2 n_k & b_5 < d. \end{cases}$$

Hence $Var(S)$ is quadra~~di~~tic ind, when d is between any two consecutive b_i's.

Let $K = 825 - TSA * q * 1.25$

$= 825 - 35,000 * 0.02 * 1.25 = 825 - 875 = -50.$

Then, we wish to find d which maximizes

$$\left(K + \frac{ES}{4}\right) \Big/ \sqrt{Var(S)} \qquad (*)$$

which can be simplified as

$$ES = 0.02 \left[22,500 + 2000 d \right] = 450 + 40d. \quad \forall \ 3 \le d \le 5$$

and $Var(S) = 0.02 * [1 - 0.02] * [44,500 + 2000 d^2] \ \forall \ 3 \le d \le 5$

Hence

$$E(S) = 450 + 40d$$

$$\forall\ 3 \le d \le 5.$$

$$Var(S) = 872.2 + 39.2d^2$$

and

$$\frac{d}{dd} E(S) = 40$$

$$\forall\ 3 \le d \le 5$$

$\left[\frac{d}{dx}\right.$ where $\left. x \text{ is } d\right]$

$$\frac{d}{dd} Var(S) = 78.4d$$

Unfortunately also

Finding the derivative of (*) w.r.t d and equating it to zero we have

$$(Var(S))\,[E(S)]' = 2\,[Var(S)]'\left(\frac{E(S)}{4} - 50\right)$$

$$\left[872.2 + 39.2d^2\right] * 40 = 2 * 78.4 * d\,[62.5 + 10d]$$

$$\Rightarrow \underline{d = 3.56}$$

In terms of dollars it will be $\underline{35{,}600}$.

$$\boxed{\text{Optimal Retention Limit } 35{,}600}$$

ⓒ $Var(S) = 1/4$ is also obvious as

$Var(X_i) = 1/12 \quad \forall i \Rightarrow Var(S) = \frac{3}{12} = 1/4$.

ⓒ $Pr(S \le 1.5) = 1/2$ by symmetry.

$Pr(S \le 1) = 1/6$ and $Pr(S \le 1/2) = \frac{(1/2)^3}{6} = \frac{1}{48}$ //

2.15

Contract Amt. in \$10,000 (b_i)	# of Contracts. (n_i)
1	80
2	35
3	25
5	15
10	5
	160.

$q \equiv 0.04.$

(a) $N = \sum_{1}^{5} N_i \qquad N_i \overset{\text{indep}}{\sim} Bin(n_i, 0.04)$

$\Rightarrow \quad N \sim Bin\left(\sum_{1}^{5} n_i, 0.04\right)$

$EN = 160 * 0.04 = 6.4$

$Var(N) = 160 * 0.04 * 0.96 = 6.144$

(b) Let X_i be $\ni$

$$\begin{tikzcd} & U(0, b_i) \\ X_i \arrow[ur, "0.04"] \arrow[dr, "0.96"'] & \\ & 0 \end{tikzcd}$$

Then $\quad E X_i = 0.04 * \frac{b_i}{2} = 0.02 * b_i$

$$Var(X_i) = 0.04 * \frac{b_i^2}{12} + 0.04 * 0.96 * \left(\frac{b_i}{2}\right)^2$$

$$= \frac{b_i^2 * 0.04}{4}\left[\frac{1}{3} + 0.96\right] = \frac{.0388}{3} b_i^2 //$$

Hence

$$Var(S) = \sum_{1}^{5} n_i * \frac{0.0388}{3} b_i^2 = \underline{\underline{17.072}}^{7} \; * 10^8$$ [change to \$1 as unit]

and $E(S) = \sum_{1}^{5} n_i * 0.02 * b_i = 7 \quad * 10^4$

$$Pr\left(S > (1+\theta) ES \right) = 0.01$$

But $Pr(S > (1+\theta) ES)$

$$= Pr\left(\frac{S - ES}{\sqrt{Var(S)}} > \frac{\theta ES}{\sqrt{Var(S)}} \right)$$

$$\overset{CLT}{\approx} 1 - \Phi\left(\frac{\theta ES}{\sqrt{Var(S)}} \right)$$

Hence $\dfrac{\theta ES}{\sqrt{Var(S)}} = \Phi^{-1}(0.99)$.

$$\Rightarrow \underline{\underline{\theta = 1.373153}}$$

Note: θ is invariant w.r.t. the unit — Can you argue as to why?

2.16

$$X_1 \xrightarrow{q = 1/6} B \sim f(y) = \begin{cases} 2(1-y) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$X_1 \xrightarrow{p = 5/6} \text{No claim.}$$

$$S = \sum_1^n X_i$$

$$EX_1 = \frac{1}{6} EB = \frac{1}{6} \int_0^1 2y(1-y) = \frac{1}{6}\left[y^2\Big|_0^1 - \frac{2}{3} y^3\Big|_0^1 \right]$$

$$= \frac{1}{6}\left[1 - \frac{2}{3}\right] = \frac{1}{18}$$

$$Var(X_1) = \underbrace{\frac{1}{6} Var(B)}_{E(Var(X_1|I))} + \underbrace{\frac{5}{36} (EB)^2}_{Var(E(X|I))}$$

$I \to$ indicator for claim.

$$Var(B) = EB^2 - (EB)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = 1/18$$

$$\Rightarrow Var(X_1) = \frac{1}{6} * \frac{1}{18} + \left(\frac{5}{36}\right) * \frac{1}{9} = \frac{8}{324} = 2/81$$

Hence $ES = 32 * \frac{1}{18} = 16/9$

$$Var(S) = 32 * \frac{2}{81} = \frac{64}{81}$$

$$\frac{S - 16/9}{\sqrt{64/81}} = \frac{S - 16/9}{8/9} \text{ is approx. } N(0,1).$$

Hence

$$\Pr(S > 4) = \Pr\left(\frac{S - 16/9}{8/9} > \frac{4 - 16/9}{8/9}\right)$$

$$\overset{CLT}{\approx} \Pr\left(N(0,1) > \frac{20/9}{8/9}\right)$$

$$= 1 - \Phi(2.5) = \underline{\underline{0.00621}}$$