

Precise Large Deviations for Sums of Random Variables with Consistently Varying Tails

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Abstract

Let $\{X_k, k \geq 1\}$ be a sequence of independent, identically distributed nonnegative random variables with common distribution function F and finite expectation $\mu > 0$. Under the assumption that the tail probability $\bar{F}(x) = 1 - F(x)$ is consistently varying as x tends to infinity, this paper investigates precise large deviations for both the partial sums S_n and the random sums $S_{N(t)}$, where $N(t)$ is a counting process independent of the sequence $\{X_k, k \geq 1\}$. The obtained results improve some related classical ones. Applications to a risk model with negatively associated claim occurrences and to a risk model with a doubly stochastic arrival process (extended Cox process) are proposed.

Keywords: Consistently varying tail; doubly stochastic process; heavy tail; Matuzewska index; negative association; precise large deviations; random sums.

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1 Introduction

Throughout this paper, $\{X, X_k, k \geq 1\}$ denotes a sequence of independent, identically distributed (i.i.d.) and nonnegative random variables with common distribution function $F(x) = 1 - \bar{F}(x) = \mathbb{P}(X \leq x)$ and finite expectation μ , where $\bar{F}(x) > 0$ for all x . For $n \geq 1$, we denote by S_n the n th partial sum of the sequence $\{X_k, k \geq 1\}$. All limit relationships, unless otherwise stated, are for $n \rightarrow \infty$ or $t \rightarrow \infty$.

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Mainstream research on precise large-deviation probabilities has been concentrated on the study of the asymptotics

$$\mathbb{P}(S_n - n\mu > x) \sim n\bar{F}(x) \quad (1.1)$$

which holds uniformly for some x -region T_n . The uniformity in (1.1) is understood in the following sense:

$$\lim_{n \rightarrow \infty} \sup_{x \in T_n} \left| \frac{\mathbb{P}(S_n - n\mu > x)}{n\bar{F}(x)} - 1 \right| = 0.$$

Some earlier work on precise large deviations can be found in Nagaev (1969_{a, b}), and Heyde (1967_{a, b}, 1968). A random variable is said to have regularly varying tail if its distribution function, F , satisfies

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha} \quad \text{for any } y > 0, \quad (1.2)$$

for some $\alpha > 1$. We use $\mathcal{R}_{-\alpha}$ to denote this class of random variables. Nagaev (1973, 1979) studied the asymptotics (1.1) for random variables with regularly varying tail. Cline and Hsing (1991) obtained (1.1) for a larger class, the so-called ERV (Extended Regularly Varying) class, and a more flexible x -region T_n . By definition, a distribution function F with a support on $[0, \infty)$ is in $\text{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ if

$$y^{-\beta} \leq \liminf_{x \rightarrow \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \leq \limsup_{x \rightarrow \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \leq y^{-\alpha} \quad \text{for any } y > 1. \quad (1.3)$$

If we restrict ourselves to the case $T_n = [\gamma n, \infty)$ for arbitrarily fixed $\gamma > 0$, we can restate their result as follows.

Proposition 1.1. (Cline and Hsing (1991).) *If F belongs to the class $\text{ERV}(-\alpha, -\beta)$ with $1 < \alpha \leq \beta < \infty$, then (1.1) holds uniformly for the x -region $T_n = [\gamma n, \infty)$ for $\gamma > 0$.*

See also Section 1 of Klüppelberg and Mikosch (1997) for this result. A nice review of recent developments in precise large deviations is given in Mikosch and Nagaev (1998). Other reviews can be found in Nagaev (1979) and Rozovski (1993). See also the monographs Vinogradov (1994), Gnedenko and Korolev (1996) and Meerschaert and Scheffler (2001).

Recently, the precise large deviations for the randomly indexed sum (random sum),

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad t \geq 0,$$

have been investigated by many researchers such as Cline and Hsing (1991), Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998) and Tang *et al.* (2001), among many others. Here, the random index $N(\cdot)$ is a nonnegative integer-valued process, independent of

the sequence $\{X_k, k \geq 1\}$. We always suppose that the mean function $\lambda(t) = \mathbb{E}N(t)$ tends to ∞ as $t \rightarrow \infty$. The following proposition, which extends Proposition 1.1 to the case of random sums, is Theorem 3.1 of Klüppelberg and Mikosch (1997); see also Proposition 7.1 of Mikosch and Nagaev (1998) for a further extension.

Proposition 1.2. (Klüppelberg and Mikosch (1997).) *If F belongs to the class $\text{ERV}(-\alpha, -\beta)$ with $1 < \alpha \leq \beta < \infty$, and $N(t)$ satisfies that*

$$\frac{N(t)}{\lambda(t)} \xrightarrow{p} 1, \quad (A)$$

and that, for some $\varepsilon > 0$ and any $\delta > 0$,

$$\sum_{k > (1+\delta)\lambda(t)} (1 + \varepsilon)^k \mathbb{P}(N(t) > k) = o(1), \quad (B)$$

then, for any $\gamma > 0$,

$$\mathbb{P}(S_{N(t)} - \mu\lambda(t) > x) \sim \lambda(t)\bar{F}(x) \quad \text{uniformly for } x \geq \gamma\lambda(t). \quad (1.4)$$

As was verified by Klüppelberg and Mikosch (1997), the homogenous Poisson process satisfies both assumptions *A* and *B*. Some applications of Proposition 1.2 to insurance and finance can be found in Chapter 8 of Embrechts *et al.* (1997) and some of the above-mentioned references.

In this paper, we aim to extend the two propositions above. The rest of the paper is organized as follows. Section 2 presents some preliminaries and introduces another useful subexponential subclass, denoted by \mathcal{C} , which consists of distribution functions with consistently varying tails. Some discussions on the Matuszewska index γ_F of a distribution function F with dominatedly varying tail are also given. In Section 3, we prove the large-deviation result (1.1) for $T_n = [\gamma n, \infty)$ and for $F \in \mathcal{C}$. In Section 4, we improve Proposition 1.2 in such a way that we not only extend the regularity of the claim-size distribution from ERV to \mathcal{C} , but also weaken Assumptions *A* and *B* to the single condition that

$$\mathbb{E}N^p(t)\mathbb{I}_{(N(t) > (1+\delta)\lambda(t))} = O(\lambda(t)), \quad (C)$$

for some $p > \gamma_F$ and any $\delta > 0$. Assumption *C* is satisfied, for example, by the renewal counting process as verified in Lemma 3.5 of Tang *et al.* (2001). Two examples are proposed in the last section as applications. We consider the case where the i.i.d. sequence $\{X_k, k \geq 1\}$ is associated with another sequence of Bernoulli variables which obey a certain kind of negative association, and the case where the counting process $N(t)$, $t \geq 0$, is specified as a so-called doubly stochastic process.

2 Preliminaries

2.1 Heavy-tailed distributions

We are interested in heavy-tailed distributions. We say a nonnegative random variable X (or its distribution function F) is heavy-tailed if it has no finite exponential moments. The most important heavy-tailed subclass is the subexponential class \mathcal{S} . By definition, a distribution function F with support on $[0, \infty)$ belongs to the class \mathcal{S} if

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$$

for any $n \geq 2$ (or, equivalently, for $n = 2$), where F^{*n} denotes the n -fold convolution of F . There is another important heavy-tailed subclass, \mathcal{D} , which is closely related to \mathcal{S} . By definition, a distribution function F with support on $[0, \infty)$ belongs to the class \mathcal{D} (is of dominatedly varying tail) if

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for any $0 < y < 1$ (or, equivalently for $y = 1/2$). For more details on heavy-tailed distributions please refer to Embrechts *et al.* (1997) and Meerschaert and Scheffler (2001).

Recently, Klüppelberg and Mikosch (1997) and Tang *et al.* (2001) considered large deviations for the random sums of random variables with distribution function belonging to the class ERV. In the present paper we will consider a slightly larger subclass of heavy-tailed distributions and name it as the class \mathcal{C} in short.

Definition 2.1. (The class \mathcal{C} .) *A distribution function F with support on $[0, \infty)$ belongs to \mathcal{C} if*

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1, \quad \text{or equivalently,} \quad \lim_{y \nearrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1. \quad (2.1)$$

Such a distribution function F is said to have a consistently varying tail.

The regularity property in (2.1) of the tail probability \overline{F} was first introduced and named ‘intermediate regular varying property’ by Cline (1994). Some closely related discussions of the class \mathcal{C} can be found in Cline and Samorodnitsky (1994). Recently, Jelenković and Lazar (1999) used the regularity property in (2.1) when they considered some problems in queueing systems and applications. Schlegel (1998) also applied the regularity property in (2.1) to study asymptotic properties for ruin probabilities in perturbed risk models.

It is easy to check that, whenever $0 < \alpha \leq \gamma \leq \beta < \infty$, we have the following inclusion relationship:

$$\mathcal{R}_{-\gamma} \subset \text{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{S}.$$

See Subsection 4.3 of Jelenković and Lazar (1999) for the last inclusion. Cline and Samorodnitsky (1994, p. 87) constructed some examples to show that \mathcal{C} is strictly larger than ERV and that $\mathcal{D} \cap \mathcal{S}$ is strictly larger than \mathcal{C} . Related discussions can also be found in Cai and Tang (2004).

Obviously, if $F \in \mathcal{D}$, then for any $y > 0$, $\overline{F}(xy)$ and $\overline{F}(x)$ are of the same order as $x \rightarrow \infty$ in the sense that

$$0 < \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty. \quad (2.2)$$

We denote this fact by $\overline{F}(xy) \asymp \overline{F}(x)$. Set

$$\gamma(y) := \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \quad \text{and} \quad \gamma_F := \inf \left\{ -\frac{\log \gamma(y)}{\log y} : y > 1 \right\}.$$

In the terminology of Bingham *et al.* (1987), γ_F is called *the upper Matuszewska index* of the nonnegative and nondecreasing function $f(x) = (\overline{F}(x))^{-1}$, $x \geq 0$. Without any danger of confusion we simply call γ_F as *the Matuszewska index* of the distribution function F . Clearly, if $F \in \text{ERV}(-\alpha, -\beta)$, then $\alpha \leq \gamma_F \leq \beta$. See Chapter 2.1 of Bingham *et al.* (1987) and Cline and Samorodnitsky (1994) for more details of the Matuszewska index.

2.2 Lemmas

We shall need the following result in the proofs of Theorems 3.1 and 4.1.

Lemma 2.2. *For a distribution function $F \in \mathcal{D}$ with a finite expectation, $1 \leq \gamma_F < \infty$ and, as $x \rightarrow \infty$,*

$$x^{-\gamma} = o(\overline{F}(x)) \quad \text{for any } \gamma > \gamma_F. \quad (2.3)$$

Proof. This is a part of Lemma 3.5 of Tang and Tsitsiashvili (2003). For the sake of completeness, we copy here their proof with slight adjustments. From Proposition 2.2.1 of Bingham *et al.* (1987) we know that, for any $\rho > \gamma_F$, there are positive constants x_0 and B such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq B(x/y)^\rho \quad (2.4)$$

holds whenever $x \geq y \geq x_0$. Hence, fixing the variable y in (2.4) leads to the result (2.3). From the inequality (2.1.9) in Theorem 2.1.8 of Bingham *et al.* (1987) we easily see that $F \in \mathcal{D}$ if and only if $\gamma_F < \infty$. Clearly, the result (2.3) implies $\mathbb{E}X_+^\gamma = \infty$ for any $\gamma > \gamma_F$. Hence, $\gamma_F \geq 1$ holds immediately. This completes the proof of Lemma 2.2.

Now we restate the inequality (2.4) as follows.

Lemma 2.3. For $F \in \mathcal{D}$ and every $\rho > \gamma_F$, there exist positive x_0 and B such that, for all $\theta \in (0, 1]$ and all $x \geq \theta^{-1}x_0$,

$$\frac{\overline{F}(\theta x)}{\overline{F}(x)} \leq B\theta^{-\rho}. \quad (2.5)$$

We remark that (2.5) holds even when $\theta = \theta(x)$ tends to 0 as $x \rightarrow \infty$. This inequality will play a key role in the proof of our Theorem 3.1.

Lemma 2.4 below is a reformulation of Lemma 3.2 of Tang *et al.* (2001); a similar result and related discussions can be found in Fuk and Nagaev (1971).

Lemma 2.4. Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. nonnegative random variables with common distribution function F and finite expectation μ . Then, for all $v > 0$, $x > 0$ and $n \geq 1$,

$$\mathbb{P}(S_n > x) \leq n\overline{F}(x/v) + (e\mu n/x)^v. \quad (2.6)$$

Lemmas 2.5 and 2.6 are reformulations of Lemmas 3.3 and 3.5 of Tang *et al.* (2001); we will need these two lemmas in the later part of this paper.

Lemma 2.5. Let $\{\zeta(t), t \geq 0\}$ be a stochastic process with a common expectation $\mathbb{E}\zeta(t) = 1$. If, for any fixed $\delta > 0$,

$$\mathbb{E}\zeta(t)\mathbb{I}_{(\zeta(t) > 1+\delta)} = o(1),$$

then $\zeta(t) \rightarrow_p 1$.

Lemma 2.6. Suppose that $\{N(t), t \geq 0\}$ is an ordinary renewal process which is driven by a sequence of i.i.d. nonnegative random variables $\{Y_n, n \geq 1\}$ with common finite expectation. Then, for any positive constants θ and p ,

$$\mathbb{E}N^p(t)\mathbb{I}_{(N(t) > (1+\theta)\mathbb{E}N(t))} = o(1).$$

3 Large deviations for nonrandom sums

Now we are in a position to establish the first main result of this paper.

Theorem 3.1. Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. nonnegative random variables with common distribution function $F \in \mathcal{C}$ and finite expectation μ . Then, for any fixed $\gamma > 0$,

$$\mathbb{P}(S_n - n\mu > x) \sim n\overline{F}(x) \quad \text{uniformly for } x \geq \gamma n. \quad (3.1)$$

Remark 3.1. The relation (3.1) is an accurate form of an inequality in Tang and Yan (2002). It is complementary to Theorem 2.1 of Cline and Hsing (1991). Compared with the latter, our result is focused on x -regions of the type $T_n = [\gamma n, \infty)$ under the weaker condition that F has a consistently varying tail.

Proof. For any $\lambda > 1$,

$$\begin{aligned}
& \mathbb{P}(S_n - n\mu > x) \\
& \geq \mathbb{P}\left(S_n - n\mu > x, \max_{1 \leq k \leq n} X_k > \lambda x\right) \\
& \geq \sum_{k=1}^n \mathbb{P}(S_n - n\mu > x, X_k > \lambda x) - \sum_{1 \leq k < l \leq n} \mathbb{P}(S_n - n\mu > x, X_k > \lambda x, X_l > \lambda x) \\
& \geq n\mathbb{P}(S_{n-1} - n\mu > (1 - \lambda)x, X_n > \lambda x) - (n\overline{F}(\lambda x))^2 \\
& = n\overline{F}(\lambda x) \left(\mathbb{P}(S_{n-1} - n\mu > (1 - \lambda)x) - n\overline{F}(\lambda x)\right). \tag{3.2}
\end{aligned}$$

Clearly, by the classical law of large numbers,

$$\liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \mathbb{P}(S_{n-1} - n\mu > (1 - \lambda)x) = 1. \tag{3.3}$$

Substituting (3.3) into (3.2) leads to

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\overline{F}(\lambda x)} & \geq \liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \left(\mathbb{P}(S_{n-1} - n\mu > (1 - \lambda)x) - n\overline{F}(\lambda x)\right) \\
& \geq \liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} (1 - n\overline{F}(\lambda x)) = 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\overline{F}(x)} & \geq \liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\overline{F}(\lambda x)} \cdot \liminf_{x \rightarrow \infty} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} \\
& \geq \liminf_{x \rightarrow \infty} \frac{\overline{F}(\lambda x)}{\overline{F}(x)}.
\end{aligned}$$

Since $F \in \mathcal{C}$ and $\lambda > 1$ is arbitrary, we conclude that

$$\liminf_{n \rightarrow \infty} \inf_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\overline{F}(x)} \geq \lim_{\lambda \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} = 1. \tag{3.4}$$

Now we start to check the upper bound for (3.1). For any $\theta \in (0, 1)$, we define

$$\tilde{X}_k := X_k \mathbb{I}_{(X_k \leq \theta x)} \quad \text{for } k \geq 1, \quad \tilde{S}_n := \sum_{k=1}^n \tilde{X}_k \quad \text{and} \quad \tilde{x} := x + n\mu.$$

By a standard truncation argument, we can show that

$$\begin{aligned}
\mathbb{P}(S_n - n\mu > x) & \leq \mathbb{P}\left(\max_{1 \leq k \leq n} X_k > \theta x\right) + \mathbb{P}\left(\max_{1 \leq k \leq n} X_k \leq \theta x, S_n - n\mu > x\right) \\
& \leq n\overline{F}(\theta x) + \mathbb{P}\left(\tilde{S}_n > \tilde{x}\right). \tag{3.5}
\end{aligned}$$

We estimate the second term in (3.5). Let $a = \max\{-\log(n\bar{F}(\theta x)), 1\}$, which tends to ∞ uniformly for $x \geq \gamma n$. For arbitrarily fixed $h > 0$,

$$\begin{aligned} \frac{\mathbb{P}\left(\tilde{S}_n > \tilde{x}\right)}{n\bar{F}(\theta x)} &\leq \frac{e^{-h\tilde{x}}\mathbb{E}e^{h\tilde{S}_n}}{n\bar{F}(\theta x)} \\ &\leq e^{-h\tilde{x}+a} \left(\int_0^{\theta x} (e^{ht} - 1)F(dt) + 1 \right)^n \\ &\leq \exp \left\{ n \int_0^{\theta x} (e^{ht} - 1)F(dt) - h\tilde{x} + a \right\}. \end{aligned} \quad (3.6)$$

The value of h above will be specified later. For any fixed $\tau > 1$, we divide the integral on the right-hand side of (3.6) into two terms as

$$\int_0^{\theta x} (e^{ht} - 1)F(dt) = \int_0^{\theta x/a^\tau} (e^{ht} - 1)F(dt) + \int_{\theta x/a^\tau}^{\theta x} (e^{ht} - 1)F(dt). \quad (3.7)$$

Applying an elementary inequality, $e^u - 1 \leq ue^u$, to the first term of the right-hand side of (3.7), we obtain that

$$\begin{aligned} \int_0^{\theta x/a^\tau} (e^{ht} - 1)F(dt) &\leq e^{h\theta x/a^\tau} \int_0^{\theta x/a^\tau} htF(dt) + e^{h\theta x/a^\tau} \bar{F}(\theta x/a^\tau) \\ &\leq h\mu e^{h\theta x/a^\tau} + e^{h\theta x/a^\tau} \bar{F}(\theta x/a^\tau). \end{aligned} \quad (3.8)$$

Substituting (3.8) into (3.6) yields that, for all large n ,

$$\begin{aligned} \frac{\mathbb{P}\left(\tilde{S}_n > \tilde{x}\right)}{n\bar{F}(\theta x)} &\leq \exp \left\{ nh\mu e^{h\theta x/a^\tau} + ne^{h\theta x/a^\tau} \bar{F}(\theta x/a^\tau) - h\tilde{x} + a \right\} \\ &\leq \exp \left\{ nh\mu e^{h\theta x/a^\tau} + Bne^{h\theta x/a^\tau} \bar{F}(\theta x) - h\tilde{x} + a \right\} \\ &= \exp \left\{ nh\mu e^{h\theta x/a^\tau} + Be^{h\theta x/a^\tau} \bar{F}(\theta x) - h\tilde{x} + a \right\}. \end{aligned} \quad (3.9)$$

Here we have used the inequality (2.5) in the second step. Letting

$$h = \frac{a - \rho\tau \log a}{\theta x}$$

in (3.9), we obtain that, for all large n ,

$$\begin{aligned} \frac{\mathbb{P}\left(\tilde{S}_n > \tilde{x}\right)}{n\bar{F}(\theta x)} &\leq \exp \left\{ nh\mu e^{(a-\rho\tau \log a)/a^\tau} + B - h(x + n\mu) + a \right\} \\ &\leq \exp \left\{ \left(e^{a^{1-\tau}} - 1 \right) hn\mu + B - hx + a \right\} \\ &= e^B \exp \left\{ o(n) \frac{a - \rho\tau \log a}{\theta x} - \frac{a - \rho\tau \log a}{\theta} + a \right\} \\ &= e^B \exp \left\{ o(a) + (1 - 1/\theta) a \right\}. \end{aligned}$$

Combining this with (3.5) gives

$$\limsup_{n \rightarrow \infty} \sup_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\bar{F}(\theta x)} \leq 1 + \limsup_{n \rightarrow \infty} \sup_{x \geq \gamma n} \frac{\mathbb{P}(\tilde{S}_n > \tilde{x})}{n\bar{F}(\theta x)} = 1.$$

Similarly to the above, by the fact $F \in \mathcal{C}$ and the arbitrariness of $\theta \in (0, 1)$ we obtain that

$$\limsup_{n \rightarrow \infty} \sup_{x \geq \gamma n} \frac{\mathbb{P}(S_n - n\mu > x)}{n\bar{F}(x)} = \lim_{\theta \nearrow 1} \limsup_{n \rightarrow \infty} \sup_{x \geq \gamma n} \left(\frac{\mathbb{P}(S_n - n\mu > x)}{n\bar{F}(\theta x)} \cdot \frac{\bar{F}(\theta x)}{\bar{F}(x)} \right) \leq 1. \quad (3.10)$$

The result (3.1) then follows from (3.4) and (3.10). \square

Remark. In the proof above, the treatment in (3.2) is similar to the proof of Theorem 4.1 of Mikosch and Nagaev (1998). The idea of the division in (3.7) is from Cline and Hsing (1991), but the method is different.

4 Large deviations for random sums

The following theorem is the second main result of this paper.

Theorem 4.1. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. nonnegative random variables with finite expectation μ and common distribution function $F \in \mathcal{C}$, independent of a nonnegative and integer-valued process $\{N(t), t \geq 0\}$. Assume that $N(\cdot)$ satisfies Assumption C. Then the precise large-deviation result (1.4) holds.*

Proof. By Lemma 2.5 with $\zeta(t) = N(t)/\lambda(t)$, we can easily see that Assumption C implies Assumption A.

We write

$$\mathbb{P}(S_{N(t)} - \mu\lambda(t) > x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n - \mu\lambda(t) > x) \mathbb{P}(N(t) = n)$$

and divide this sum into three parts as

$$\sum_{n < (1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t) \leq n \leq (1+\delta)\lambda(t)} + \sum_{n > (1+\delta)\lambda(t)} = K_1 + K_2 + K_3, \quad (4.1)$$

where $0 < \delta < \gamma/\mu$ is arbitrary. By the same approach as used in the proof of Lemma 4.2 of Klüppelberg and Mikosch (1997), we know that

$$K_1 = o(\lambda(t)\bar{F}(x)). \quad (4.2)$$

Now we deal with K_2 . By Assumption A and Theorem 3.1, we have

$$\begin{aligned}
K_2 &\leq \mathbb{P}(S_{[(1+\delta)\lambda(t)]} - \mu\lambda(t) > x) \mathbb{P}((1-\delta)\lambda(t) \leq N(t) \leq (1+\delta)\lambda(t)) \\
&\sim \mathbb{P}(S_{[(1+\delta)\lambda(t)]} - [(1+\delta)\lambda(t)]\mu > x + \mu\lambda(t) - [(1+\delta)\lambda(t)]\mu) \\
&\sim (1+\delta)\lambda(t)\bar{F}(x + \mu\lambda(t) - [(1+\delta)\lambda(t)]\mu) \\
&\leq (1+\delta)\lambda(t)\bar{F}((1 - (\delta\mu)/\gamma)x).
\end{aligned}$$

By the same treatment we obtain the corresponding asymptotic lower bound for K_2 as

$$(1-\delta)\lambda(t)\bar{F}((1 + (\delta\mu)/\gamma)x).$$

Thus, it follows from (2.1) that

$$\lim_{\delta \searrow 0} \limsup_{t \rightarrow \infty} \sup_{x \geq \gamma\lambda(t)} \left| \frac{K_2}{\lambda(t)\bar{F}(x)} - 1 \right| = 0. \quad (4.3)$$

Finally, we estimate K_3 . Letting v in (2.6) equal to the number $p > \gamma_F \geq 1$, we obtain that

$$\begin{aligned}
K_3 &\leq \sum_{n \geq (1+\delta)\lambda(t)} \mathbb{P}(S_n > x) \mathbb{P}(N(t) = n) \\
&\leq \sum_{n \geq (1+\delta)\lambda(t)} (n\bar{F}(x/p) + (e\mu n/x)^p) \mathbb{P}(N(t) = n) \\
&\asymp \bar{F}(x)\mathbb{E}N(t)\mathbb{I}_{(N(t) \geq (1+\delta)\lambda(t))} + x^{-p}\mathbb{E}N^p(t)\mathbb{I}_{(N(t) \geq (1+\delta)\lambda(t))} \\
&= o(\bar{F}(x)\mathbb{E}N^p(t)\mathbb{I}_{(N(t) \geq (1+\delta)\lambda(t))}) + x^{-p}\mathbb{E}N^p(t)\mathbb{I}_{(N(t) \geq (1+\delta)\lambda(t))} \\
&= o(\lambda(t)\bar{F}(x)), \tag{4.4}
\end{aligned}$$

where, in the last step, we have used Assumption C and Lemma 2.2. Substituting (4.2), (4.3) and (4.4) into (4.1), we know that (1.4) holds. \square

5 Applications of Theorem 4.1

In this section, we provide two examples of applications of Theorem 4.1. In these examples, our Assumption C is fulfilled, whereas the related assumptions in the literature are not.

5.1 Negatively associated claim occurrences

In the context of insurance risk theory, the claimsizes X_k , $k \geq 1$, are often assumed to be i.i.d. nonnegative random variables with common distribution function F . Their occurrence times σ_k , $k \geq 1$, constitute an ordinary renewal counting process

$$N(t) = \sup\{n : \sigma_n \leq t\}, \quad t \geq 0,$$

with $\lambda(t) = \mathbb{E}N(t) < \infty$ for any $t \geq 0$. With $\sigma_0 = 0$, we can write the interarrival times by $Y_k = \sigma_k - \sigma_{k-1}$ for $k \geq 1$, which therefore form a sequence of i.i.d. nonnegative random variables. These are standard assumptions in the ordinary renewal model.

We consider the model of Denuit *et al.* (2002). In this model, the k th insurance policy, $k \geq 1$, is associated with a Bernoulli variable I_k . The variable I_k has a common expectation q , where $q \in (0, 1]$ and q is the claim-occurrence probability of the k th policy, $k \geq 1$. The total claim amount up to time t is then

$$S_1(t) = \sum_{k=1}^{N(t)} X_k I_k, \quad t \geq 0. \quad (5.1)$$

We assume that the three sequences $\{X_k, k \geq 1\}$, $\{Y_k, k \geq 1\}$ and $\{I_k, k \geq 1\}$ are mutually independent and that the sequence $\{I_k, k \geq 1\}$ is negatively associated. Therefore, $S_1(t)$ is a nonstandard random sum unless the parameter q is equal to 1. Generally speaking, a sequence $\{I_k, k \geq 1\}$ is said to be negatively associated (NA) if, for any disjoint finite subsets A and B of $\{1, 2, \dots\}$ and any coordinatewise monotonically increasing functions f and g , the inequality

$$\text{Cov}[f(I_i; i \in A), g(I_j; j \in B)] \leq 0$$

holds whenever the moment involved exists. For details about the notion of NA, please refer to Alam and Saxena (1981), Joag-Dev and Proschan (1983) and Shao (2000), among others.

In this subsection, we will show that the precise large-deviation result obtained in Section 4 can be applied to the random sum $S_1(t)$ defined in (5.1). Write

$$N^*(t) = \sup\{n : \sigma_n \leq t \text{ and } I_n = 1\}, \quad t \geq 0,$$

which represents the number of claims that really occur in the interval $[0, t]$. Clearly,

$$N^*(t) = \sum_{k=1}^{N(t)} I_k, \quad \mathbb{E}N^*(t) = \mathbb{E} \sum_{k=1}^{N(t)} I_k = q\mathbb{E}N(t) = q\lambda(t), \quad t \geq 0. \quad (5.2)$$

The random sum $S_1(t)$ can be rewritten as

$$S_1(t) = \sum_{k: 1 \leq k \leq N(t), I_k=1} X_k \stackrel{d}{=} \sum_{k=1}^{N^*(t)} X_k, \quad (5.3)$$

where $\stackrel{d}{=}$ denotes equality in distribution. We notice that the right-hand side of (5.3) is a standard random sum, i.e. that the random index $N^*(t)$ is independent of the summands $\{X_k, k \geq 1\}$. Based on (5.3), in order to establish precise large deviations for the random sum $S_1(t)$, it suffices to check Assumption C for the counting process $N^*(t)$, $t \geq 0$.

We introduce another i.i.d. sequence $\{I_k^\perp, k \geq 1\}$, independent of the process $\{N(t), t \geq 0\}$, such that $I_1^\perp \stackrel{d}{=} I_1$. For $x > 0$ and $p > 1$, we define a function as follows:

$$f(u|x, p) = \begin{cases} 0, & u \leq (1 - 1/p)x, \\ px^{p-1}(u - (1 - 1/p)x), & (1 - 1/p)x < u \leq x, \\ u^p, & u > x. \end{cases} \quad (5.4)$$

Clearly, the function $f(u|x, p)$ is nonnegative, increasing and convex in $u \in (-\infty, \infty)$, and for any $x > 0$ and $p > 1$,

$$u^p \mathbb{I}_{(u>x)} \leq f(u|x, p) \leq u^p \mathbb{I}_{(u>(1-1/p)x)}. \quad (5.5)$$

By Theorem 1 of Shao (2000) we know that the inequality

$$\mathbb{E}g\left(\sum_{k=1}^n I_k\right) \leq \mathbb{E}g\left(\sum_{k=1}^n I_k^\perp\right)$$

holds for any $n \geq 1$ and any convex function g on $(-\infty, \infty)$, whenever the expectation on the right-hand side exists. By this result and the inequalities in (5.5) we obtain that, for any positive θ and p ,

$$\begin{aligned} \mathbb{E}(N^*(t))^p \mathbb{I}_{(N^*(t)>(1+\theta)\mathbb{E}N^*(t))} &\leq \mathbb{E}f(N^*(t)|(1+\theta)\mathbb{E}N^*(t), p) \\ &= \mathbb{E}\left\{\mathbb{E}\left(f\left(\sum_{k=1}^{N(t)} I_k \middle| (1+\theta)\mathbb{E}N^*(t), p\right) \middle| N(t)\right)\right\} \\ &\leq \mathbb{E}\left\{\mathbb{E}\left(f\left(\sum_{k=1}^{N(t)} I_k^\perp \middle| (1+\theta)\mathbb{E}N^*(t), p\right) \middle| N(t)\right)\right\} \\ &= \mathbb{E}f\left(\sum_{k=1}^{N(t)} I_k^\perp \middle| (1+\theta)\mathbb{E}N^*(t), p\right) \\ &\leq \mathbb{E}\left(\sum_{k=1}^{N(t)} I_k^\perp\right)^p \mathbb{I}_{(\sum_{k=1}^{N(t)} I_k^\perp > (1-1/p)(1+\theta)\mathbb{E}N^*(t))}. \end{aligned}$$

For any $\theta > 0$, we choose $p > 1$ sufficiently large that

$$\theta(p) = (1 - 1/p)(1 + \theta) - 1 > 0. \quad (5.6)$$

This gives

$$\mathbb{E}(N^*(t))^p \mathbb{I}_{(N^*(t)>(1+\theta)\mathbb{E}N^*(t))} \leq \mathbb{E}\left(\tilde{N}(t)\right)^p \mathbb{I}_{(\tilde{N}(t)>(1+\theta(p))\mathbb{E}\tilde{N}(t))} \quad (5.7)$$

with $\tilde{N}(t) = \sum_{k=1}^{N(t)} I_k^\perp$. We further introduce $\nu_0 = 0$ and

$$\nu_n = \inf\{k > \nu_{n-1} : I_k^\perp = 1\}, \quad \tilde{Y}_n = \sum_{k=\nu_{n-1}}^{\nu_n} Y_k, \quad n \geq 1.$$

It is not difficult to see that $\left\{ \left((\nu_n - \nu_{n-1}), \widetilde{Y}_n \right), n \geq 1 \right\}$ forms a sequence of i.i.d. random pairs, and that ν_1 is a geometric random variable with parameter $q \in (0, 1]$. Moreover,

$$\widetilde{N}(t) = \sup \left\{ n : \sum_{k=1}^n \widetilde{Y}_k \leq t \right\}, \quad t \geq 0,$$

which means that the process $\{\widetilde{N}(t), t \geq 0\}$ is also an ordinary renewal counting process driven by the sequence of i.i.d. nonnegative random variables $\{\widetilde{Y}_n, n \geq 1\}$. If we assume that the interarrival time Y_1 has a finite expectation, then

$$\mathbb{E} \widetilde{Y}_1 = \mathbb{E} \sum_{k=1}^{\nu_1} Y_k = \mathbb{E} \nu_1 \cdot \mathbb{E} Y_1 = \frac{\mathbb{E} Y_1}{q} < \infty.$$

Thus, by Lemma 2.6 we immediately obtain that the right-hand side of (5.7) equals $O\left(\mathbb{E} \widetilde{N}(t)\right)$. Then, it follows from this and (5.2) that, for any $\theta > 0$ and all large $p > 0$,

$$\mathbb{E} (N^*(t))^p \mathbb{I}_{(N^*(t) > (1+\theta)\mathbb{E} N^*(t))} = O\left(\mathbb{E} \widetilde{N}(t)\right) = O\left(\mathbb{E} N^*(t)\right).$$

Hence, the counting process $N^*(t)$ satisfies Assumption C .

Then, by (5.3) and Theorem 4.1, we know that, if the claim-size distribution F is in \mathcal{C} with a finite expectation, then, for any $\gamma > 0$ and all $x \geq \gamma\lambda(t)$,

$$\mathbb{P}(S_1(t) - \mu q \lambda(t) > x) \sim \mathbb{P}\left(\sum_{k=1}^{N^*(t)} X_k - \mu \mathbb{E} N^*(t) > x\right) \sim q \lambda(t) \overline{F}(x). \quad (5.8)$$

We summarize the above in the following proposition.

Proposition 5.1. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. nonnegative random variables with finite expectation μ and common distribution function $F \in \mathcal{C}$, let $\{I_k, k \geq 1\}$ be an NA sequence of Bernoulli variables with common expectation $q \in (0, 1]$, and let $\{N(t), t \geq 0\}$ be an ordinary renewal counting process driven by a sequence of i.i.d. nonnegative random variables $\{Y_k, k \geq 1\}$ which have common finite expectation. Suppose that the sequences $\{X_k, k \geq 1\}$, $\{I_k, k \geq 1\}$ and the process $\{N(t), t \geq 0\}$ are mutually independent. Then, for any $\gamma > 0$ and all $x \geq \gamma\lambda(t)$,*

$$\mathbb{P}\left(\sum_{k=1}^{N(t)} X_k I_k - \mu q \lambda(t) > x\right) \sim q \lambda(t) \overline{F}(x).$$

5.2 Doubly stochastic counting process

Let $N_1(t), t \geq 0$, be an ordinary renewal process which is generated by i.i.d. nonnegative random variables $\{Y_k, k \geq 1\}$ with $\mathbb{E} Y_1 = 1$, and let $\Lambda(t), t \geq 0$, be another right-continuous

nondecreasing process with $\Lambda(0) = 0$, independent of $N_1(t)$, and $\mathbb{P}(\Lambda(t) < \infty) = 1$ for any $t \geq 0$.

We now concentrate our interests on a doubly stochastic process $N(t)$ which is defined as the composition of $N_1(t)$ and $\Lambda(t)$:

$$N(t) = N_1(\Lambda(t)), \quad t \geq 0. \quad (5.9)$$

When $N_1(t)$, $t \geq 0$, is a homogeneous Poisson process with unit intensity, the doubly stochastic process (5.9) is called a Cox process. For an overview on Cox processes and their applications to actuarial and financial mathematics, we refer the reader to Grandell (1976) and Bening and Korolev (2002); see also Korolev (1999, 2001).

In this subsection we prove that the precise large-deviation result obtained in Theorem 4.1 can be applied to the compound process

$$S_2(t) = \sum_{k=1}^{N(t)} X_k, \quad t \geq 0, \quad (5.10)$$

where $\{X_k, k \geq 1\}$ is a sequence of i.i.d. nonnegative random variables with common distribution function F , independent of the doubly counting process $N(t)$. Suppose that $F \in \mathcal{C}$ has a finite expectation. According to Theorem 4.1, it suffices to check Assumption C for the doubly stochastic process $N(t)$.

We assume that the inner stochastic process $\Lambda(t)$ satisfies the conditions that $\lambda^*(t) := \mathbb{E}\Lambda(t) < \infty$ for any $t \geq 0$ but $\lambda^*(t) \rightarrow \infty$, and that, for some $p > \gamma_F$ and any $\theta > 0$,

$$\mathbb{E}\Lambda^p(t)\mathbb{I}_{(\Lambda(t) > (1+\theta)\lambda^*(t))} = O(\mathbb{E}\Lambda(t)), \quad (5.11)$$

where γ_F denotes the Matuszewska index of the distribution function F . Recall Lemma 2.2. We know that $\gamma_F \geq 1$. By virtue of Lemma 2.5, the assumption (5.11) indicates that

$$\frac{\Lambda(t)}{\lambda^*(t)} \xrightarrow{p} 1.$$

Since, in the structure of $N(t) = N_1(\Lambda(t))$, $t \geq 0$, the ordinary renewal process $N_1(t)$ is generated by the sequence $\{Y_k, k \geq 1\}$, which is independent of the stochastic process $\Lambda(t)$, $t \geq 0$, we have

$$\mathbb{E}(N(t) | \Lambda(t)) = \mathbb{E}N_1(x)|_{x=\Lambda(t)}. \quad (5.12)$$

Noting that $\mathbb{E}N_1(t) \sim t$, based on the understanding in (5.12), we obtain that

$$\lambda(t) = \mathbb{E}N(t) = \mathbb{E}(\mathbb{E}(N(t) | \Lambda(t))) \sim \lambda^*(t). \quad (5.13)$$

Now we claim that

$$\frac{\mathbb{E}(N(t) | \Lambda(t))}{\Lambda(t)} \xrightarrow{p} 1. \quad (5.14)$$

Here, in the case where $\Lambda(t) = 0$, we assume that the left-hand side of (5.14) equals 1. In fact, for arbitrarily given $\varepsilon > 0$ we can find an $M > 0$ large enough that, for $t > M$,

$$(1 - \varepsilon)t \leq \mathbb{E}N_1(t) \leq (1 + \varepsilon)t.$$

It follows from (5.12) that

$$(1 - \varepsilon)\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)} \leq \mathbb{E}(N(t)\mathbb{I}_{(\Lambda(t) > M)} | \Lambda(t)) \leq (1 + \varepsilon)\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)}.$$

Since $\Lambda(t) \xrightarrow{p} \infty$, without loss of generality we assume that $\mathbb{P}(\Lambda(t) > 0) = 1$ for all large $t > 0$ in proving (5.14). Hence, for all sufficiently large $t > 0$,

$$\begin{aligned} \frac{\mathbb{E}(N(t) | \Lambda(t))}{\Lambda(t)} &= \frac{\mathbb{E}(N(t) | \Lambda(t)) \mathbb{I}_{(\Lambda(t) > M)} + \mathbb{E}(N(t) | \Lambda(t)) \mathbb{I}_{(\Lambda(t) \leq M)}}{\Lambda(t)} \\ &\leq \frac{(1 + \varepsilon)\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)} + \mathbb{E}(N(t) | \Lambda(t)) \mathbb{I}_{(\Lambda(t) \leq M)}}{\Lambda(t)} \\ &\leq (1 + \varepsilon)\mathbb{I}_{(\Lambda(t) > M)} + \frac{\mathbb{E}(N_1(M))}{\Lambda(t)} \xrightarrow{p} 1 + \varepsilon. \end{aligned}$$

The corresponding lower bound can similarly be derived as

$$\frac{\mathbb{E}(N(t) | \Lambda(t))}{\Lambda(t)} \geq \frac{\mathbb{E}(N(t) | \Lambda(t)) \mathbb{I}_{(\Lambda(t) > M)}}{\Lambda(t)} \geq \frac{(1 - \varepsilon)\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)}}{\Lambda(t)} \xrightarrow{p} 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the claimed result (5.14) immediately.

For the p and θ in (5.11), we arbitrarily choose θ_1 and θ_2 such that $0 < \theta_2 < \theta_1 < \theta$. It follows from (5.13) that, for all sufficiently large $t > 0$,

$$\begin{aligned} &\mathbb{E}N^p(t)\mathbb{I}_{(N(t) > (1+\theta)\lambda(t))} = \mathbb{E}\left(\mathbb{E}\left(N^p(t)\mathbb{I}_{(N(t) > (1+\theta)\lambda(t))} | \Lambda(t)\right)\right) \\ &\leq \mathbb{E}\left(\mathbb{E}\left(N^p(t) \cdot \mathbb{I}_{(N(t) > (1+\theta_1)\lambda^*(t))} \cdot \left(\mathbb{I}_{(\Lambda(t) > (1+\theta_2)\lambda^*(t))} + \mathbb{I}_{(\Lambda(t) \leq (1+\theta_2)\lambda^*(t))}\right) | \Lambda(t)\right)\right) \\ &: = J_1 + J_2. \end{aligned} \tag{5.15}$$

Recall that the ordinary renewal process $N_1(t)$ is generated by the sequence $\{Y_k, k \geq 1\}$. Both of the following relations are immediate consequences of Lemma 2.6: for any $p > 0$, $\theta > 0$,

$$\mathbb{E}N_1^p(t) = O(t^p), \quad C(p, \theta) := \sup_{t \geq 0} \{\mathbb{E}N_1^p(t)\mathbb{I}_{(N_1(t) > (1+\theta)\mathbb{E}N_1(t))}\} < \infty. \tag{5.16}$$

We will need these two relations in the following steps.

Now we continue to check Assumption C for the doubly stochastic process $N(t)$. By (5.16), we can find some constant $C > 0$ such that, for all large $t > 0$,

$$\begin{aligned} J_1 &= \mathbb{E}\left(\mathbb{E}\left(N^p(t) \cdot \mathbb{I}_{(N(t) > (1+\theta_1)\lambda^*(t))} \cdot \mathbb{I}_{(\Lambda(t) > (1+\theta_2)\lambda^*(t))} | \Lambda(t)\right)\right) \\ &\leq \mathbb{E}\left(\mathbb{E}\left(N^p(t)\mathbb{I}_{(\Lambda(t) > (1+\theta_2)\lambda^*(t))} | \Lambda(t)\right)\right) \\ &= \mathbb{E}\left(\mathbb{I}_{(\Lambda(t) > (1+\theta_2)\lambda^*(t))}\mathbb{E}\left(N_1^p(\Lambda(t)) | \Lambda(t)\right)\right) \\ &\leq C\mathbb{E}\Lambda^p(t)\mathbb{I}_{(\Lambda(t) > (1+\theta_2)\lambda^*(t))} = O(\lambda(t)), \end{aligned} \tag{5.17}$$

where, in the last step, we used the assumption (5.11) and the result (5.13). As for J_2 , with $\theta_3 = (1 + \theta_1)/(1 + \theta_2) - 1 > 0$, we have

$$\begin{aligned} J_2 &= \mathbb{E} \left(\mathbb{E} \left(N^p(t) \cdot \mathbb{I}_{(N(t) > (1+\theta_1)\lambda^*(t))} \cdot \mathbb{I}_{(\Lambda(t) \leq (1+\theta_2)\lambda^*(t))} \mid \Lambda(t) \right) \right) \\ &\leq \mathbb{E} \left(\mathbb{E} \left(N^p(t) \mathbb{I}_{(N(t) > (1+\theta_3)\Lambda(t))} \mid \Lambda(t) \right) \right). \end{aligned}$$

Similarly to (5.12), we have

$$\mathbb{E} \left(N^p(t) \mathbb{I}_{(N(t) > (1+\theta_3)\Lambda(t))} \mid \Lambda(t) \right) = \mathbb{E} \left(N_1^p(x) \mathbb{I}_{(N_1(x) > (1+\theta_3)x)} \mid x = \Lambda(t) \right).$$

From this and the second relation in (5.16) we know that

$$\sup_{t \geq 0} J_2 < \infty. \quad (5.18)$$

Substituting (5.17) and (5.18) into (5.15), we finally obtain

$$\mathbb{E} N^p(t) \mathbb{I}_{(N(t) > (1+\theta)\lambda(t))} \leq J_1 + J_2 = O(\lambda(t)).$$

This means that the doubly stochastic process $N(t)$ satisfies Assumption *C*.

We summarize the above in the following proposition.

Proposition 5.2. *Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. nonnegative random variables with common distribution function $F \in \mathcal{C}$ and finite expectation μ , and let $\{N(t), t \geq 0\}$ be a doubly stochastic process defined by (5.9) with $\{\Lambda(t), t \geq 0\}$ satisfying (5.11). If the sequence $\{X_k, k \geq 1\}$ and the process $\{N(t), t \geq 0\}$ are mutually independent, then the random sum (5.10) satisfies the precise large-deviation result: for any $\gamma > 0$, the relation*

$$\mathbb{P}(S_2(t) - \mu\lambda^*(t) > x) \sim \lambda^*(t) \overline{F}(x) \quad (5.19)$$

holds uniformly for $x \geq \gamma\lambda^(t)$.*

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