Finite and Infinite Time Ruin Probabilities in the Presence of Stochastic Returns on Investments

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Abstract

This paper investigates the finite and infinite time ruin probabilities in a discrete time stochastic economic environment. Under the assumption that the insurance risk - the total net loss within one time period - is extended-regularly-varying or rapidly-varying tailed, various precise estimates for the ruin probabilities are derived. In particular, some estimates obtained are uniform with respect to the time horizon, hence apply for the case of infinite time ruin.

Keywords: Asymptotics; class $\mathcal{S}(\gamma)$; endpoint; extended regular variation; financial risk; insurance risk; rapid variation; ruin probability

1 Introduction

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with generic random variable X, let $\{Y_n, n = 1, 2, \ldots\}$ be another sequence of i.i.d. and positive random variables with generic random variable Y, and let the two sequences be mutually independent. In this paper we are interested in the tail probabilities of the quantities

$$U_n = \max_{0\le k\le n} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j, \qquad n = 1, 2, \ldots, \tag{1.1}$$

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and

$$U_\infty = \max_{0\le k<\infty} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j, \tag{1.2}$$

where $\sum_{i=1}^{0}(\cdot) = 0$ by convention. Clearly, $0 \le U_\infty \le \sum_{i=1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j$. It is well known that the right-hand side converges almost surely if $-\infty \le \mathbb{E}\log Y < 0$ and $\mathbb{E}\log^+ X^+ < \infty$; see Vervaat (1979, Theorem 1.6) and Brandt (1986, Theorem 1). Therefore, the maximum U_∞ has a proper distribution function on $[0, \infty)$.

Although this topic is interesting in many fields of applied probability, we will restrict our discussions to ruin theory.

Following the works by Nyrhinen (1999, 2001) and Tang and Tsitsiashvili (2003), we consider a stochastic economic environment. In this environment an insurer invests his surplus into both risk-free and risky assets, which may lead to negative returns. The X_n denotes the insurer's net loss - the total claim amount minus the total incoming premium - within period n and the Y_n denotes the discount factor from time n to time $n-1$, $n = 1, 2, \ldots$. In the terminology of Norberg (1999), we call the random variable X the insurance risk and the random variable Y the financial risk. It is natural to assume $\mathbb{P}(0 < Y < \infty) = 1$.

We are concerned with the ruin probabilities of this discrete time risk model. Let $x \ge 0$ be the initial surplus. Write $A_n = -X_n$ and $R_n = Y_n^{-1} - 1$, $n = 1, 2, \ldots$. Then, A_n denotes the total net income and R_n denotes the total stochastic return rate within period n. We tacitly assume that the income A_n is calculated at time n. Hence, the surplus of the company accumulated till time n, denoted by S_n, can be characterized by

$$S_0 = x, \quad S_n = x\prod_{j=1}^{n}(1+R_j) + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n}(1+R_j), \qquad n = 1, 2, \ldots,$$

where $\prod_{j=n+1}^{n}(\cdot) = 1$ by convention. The probabilities of ruin within finite time and of ultimate ruin are defined by

$$\psi(x, n) = \mathbb{P}\left(\min_{0\le k\le n} S_k < 0\right), \qquad n = 1, 2, \ldots, \tag{1.3}$$

and

$$\psi(x) = \lim_{n\to\infty} \psi(x, n) = \mathbb{P}\left(\min_{0\le n<\infty} S_n < 0\right), \tag{1.4}$$

respectively.

We remark that there are some nontrivial cases in which the ultimate ruin probability $\psi(x) \equiv 1$ for $x \ge 0$. Actually, using the proof of Theorem 1 of Tsitsiashvili (2002) with some simple adjustments, we can prove that $\psi(x) \equiv 1$ for $x \ge 0$ if, for example, $\mathbb{E}R_1 < 0$,

$\mathbb{E}A_1^+ < \infty$, and $\mathbb{P}\left(A_1 < -\mathbb{E}A_1^+/\mathbb{E}R_1\right) > 0$, where x^+ denotes $\max\{x, 0\}$. Hence, in these cases only the finite time ruin probability needs further investigation.

In the model above, the quantity U_n defined by (1.1) describes the maximum of the discounted losses of the insurer by time n, $n = 1, 2, \ldots$, and the quantity U_∞ defined by (1.2) describes the ultimate maximum of the discounted losses. As done by Tang and Tsitsiashvili (2003), in terms of these maxima, the ruin probabilities (1.3) and (1.4) can be rewritten as

$$\psi(x, n) = \mathbb{P}\left(U_n > x\right) \qquad \text{and} \qquad \psi(x) = \mathbb{P}\left(U_\infty > x\right),$$

respectively.

Under some general conditions, Nyrhinen (1999, 2001) investigated the asymptotic behavior of the ruin probabilities and obtained large-deviation type results. Write

$$w = \sup\left\{t : \mathbb{E}Y^t \le 1\right\} \qquad \text{and} \qquad t_0 = \sup\left\{t : \mathbb{E}Y^t < \infty, \mathbb{E}\left|X\right|^t < \infty\right\}. \tag{1.5}$$

Suppose $0 < w < t_0 \le \infty$ and $\mathbb{P}(X > 0) > 0$. Directly applying Theorem 2 of Nyrhinen (2001) to the model introduced above, the relation

$$\ln \psi(x, n \ln x) \sim -R(n) \ln x \tag{1.6}$$

holds for all integers $n \ge x_0$ for some $x_0 \ge 0$, where $R(\cdot)$ is a positive function determined by the distribution of Y. Moreover, it holds that

$$\ln \psi(x) \sim -w \ln x. \tag{1.7}$$

Here and throughout, all limiting relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$ satisfying

$$l_1 = \liminf_{x\to\infty} \frac{a(x)}{b(x)} \le \limsup_{x\to\infty} \frac{a(x)}{b(x)} = l_2$$

for some $0 \le l_1 \le l_2 \le \infty$, we write $a(x) = O\left(b(x)\right)$ if $l_2 < \infty$, $a(x) = o\left(b(x)\right)$ if $l_2 = 0$, and $a(x) \asymp b(x)$ if $0 < l_1 \le l_2 < \infty$; we write $a(x) \lesssim b(x)$ if $l_2 = 1$, $a(x) \gtrsim b(x)$ if $l_1 = 1$, and $a(x) \sim b(x)$ if both $l_1 = 1$ and $l_2 = 1$.

Combined with Theorem 6.3 of Goldie (1991), relation (1.7) implies that the stronger relation

$$\psi(x) \sim Cx^{-w} \tag{1.8}$$

holds with some positive constant C. Unfortunately, the representation of this constant is too involved and ambiguous.

Similar results to (1.8) with implicit coefficients were given by Kalashnikov and Norberg (2002, Theorem 3), Frolova et al. (2002, Theorem 1(i)), and Paulsen (2002, Theorem

3.2(b)), who studied the problem in bivariate Lévy driven risk processes. Their investigations confirm that the ultimate ruin probability decreases at a power rate as the initial surplus increases. In a certain special case, Paulsen (2002, Proposition 4.1) obtained an explicit asymptotic estimate for the infinite time ruin probability. However, as recently stated in Cai and Tang (2004) and their communications with Professor Paulsen, at the end of the proof of his Proposition 4.1, the argument of applying Proposition 3.2 of Klüppelberg and Stadtmüller (1998) is not valid.

Under the standard assumptions above, Tang and Tsitsiashvili (2003) showed that

$$U_n =^d V_n, \qquad n = 1, 2, \ldots, \tag{1.9}$$

where $=^d$ denotes equality in distribution and the V_n's are determined by a Markov chain

$$V_0 = 0, \quad V_n = Y_n \max\{0,\ X_n + V_{n-1}\}, \qquad n = 1, 2, \ldots. \tag{1.10}$$

Based on formulae (1.9) and (1.10), a 'precise' - as distinct from 'the large-deviation type' as that of (1.6) and (1.7) - estimate for the finite time ruin probability was obtained for the case where the insurance risk X is dominatedly varying tailed.

We say that a distribution F is dominatedly varying tailed (or has a dominated variation), denoted by $F \in \mathcal{D}$, if

$$\limsup_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} < \infty \tag{1.11}$$

for some (or equivalently, for all) $\theta \in (0, 1)$, where $\overline{F} = 1 - F$.

However, an obvious disadvantage of the study of Tang and Tsitsiashvili (2003) is that restriction (1.11) excludes many popular distributions such as the lognormal-like, the Weibull-like, the exponential-like, and the generalized inverse Gaussian distributions, which are often applied to model the claim size distributions in ruin theory; see, for example, Asmussen (1998).

In the present paper we continue the investigation on the finite and infinite time ruin probabilities. We consider the cases where the distribution of the insurance risk X has an extended regular variation and a rapid variation, respectively. Admittedly, the latter is a more difficult case. For these cases we derive various precise asymptotic estimates for the ruin probabilities $\psi(x, n)$ and $\psi(x)$. In particular, some asymptotics obtained are uniform with respect to $n = 1, 2, \ldots$.

The rest of this paper is organized as follows. Section 2 recollects preliminaries of some well-known distribution classes, Section 3 establishes some uniform estimates for the ruin probabilities for the case where the insurance risk is extended regularly varying tailed, and Section 4 considers the case where the insurance risk is rapidly varying tailed and the financial risk is bounded or unbounded. Some lemmas that are used in establishing the main results are placed in the Appendix.

2 Some distribution classes

Throughout, for two independent random variables X and Y distributed by F and G, we denote by $F * G$ the distribution of $X + Y$ and by $F \otimes G$ the distribution of XY. In addition, we write $F^{*2} = F * F$, $F^{\otimes 2} = F \otimes F$, and so on. Whenever we mention a distribution F belonging to a certain class specified below, it always satisfies $\overline{F}(x) > 0$ for all $x \in (-\infty, \infty)$.

We say that a distribution F belongs to the class $\mathcal{R}$ if there is some $\alpha \geq 0$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} = \theta^{-\alpha} \qquad \text{for all } \theta > 0.$$

In this case we call α the (regularity) index of the distribution F and we write $F \in \mathcal{R}_{-\alpha}$.

Now we introduce a new class below, which complements the class $\mathcal{R}$ with an extreme case of $\alpha = +\infty$.

Definition 2.1. *A distribution F is said to be rapidly varying tailed (to have a rapid variation), denoted by $F \in \mathcal{R}_{-\infty}$, if*

$$\lim_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} = 0 \qquad \textit{for all } \theta > 1.$$

This property has been investigated in the literature; we refer the reader to the monographs de Haan (1970, Chapter 1.2), Bingham et al. (1987, Chapter 2.4), and Geluk and de Haan (1987).

Trivially, if $F(\cdot) \in \mathcal{R}_{-\infty}$, then $F(\cdot/c) \in \mathcal{R}_{-\infty}$ for any $c > 0$; if $\overline{F}_1(x) \asymp \overline{F}_2(x)$, then $F_1 \in \mathcal{R}_{-\infty}$ whenever $F_2 \in \mathcal{R}_{-\infty}$. For a distribution $F \in \mathcal{R}_{-\infty}$, from Theorem 1.2.2 of de Haan (1970) we know that there are positive functions $b(\cdot)$ and $c(\cdot)$ with $b(x) \to \infty$ and $c(x) \to c_0 \in (0, \infty)$ such that

$$\overline{F}(x) = c(x) \exp\left\{ -\int_1^x \frac{b(u)}{u} \mathrm{d}u \right\}, \qquad x \geq 1;$$

see also Theorem A3.12 of Embrechts et al. (1997). By this representation we easily check that for any $\varepsilon > 0$ and $K > 0$, there is some $D > 0$ such that the inequality

$$\frac{\overline{F}(x)}{\overline{F}(y)} \leq (1 + \varepsilon)\,(x/y)^{-K} \tag{2.1}$$

holds whenever $x \geq y \geq D$.

A significant subclass of $\mathcal{R}_{-\infty}$ is the generalized exponential class $\mathcal{L}(\gamma)$ with $\gamma > 0$, as defined below.

Definition 2.2. *A distribution F is said to belong to the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ if*

1. $\lim_{x\to\infty} \overline{F}^{*2}(x)/\overline{F}(x) = 2c < \infty$
2. $\lim_{x\to\infty} \overline{F}(x-t)/\overline{F}(x) = e^{\gamma t}$ *for all* $t \in (-\infty, \infty)$.

F is said to belong to the class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ if it satisfies item 2.

Classical works on these classes with applications can be found in Chistyakov (1964), Chover et al. (1973a,b), and Teugels (1975), among many others. It has been proved that, for any distribution $F \in \mathcal{S}(\gamma)$ with $\gamma \geq 0$,

$$c = \int_{-\infty}^{\infty} \exp\{\gamma x\} F(\mathrm{d}x) < \infty;$$

see Rogozin and Sgibnev (1999), Rogozin (1999), and references therein. We remark that the convergence in item *2.* is uniform for t in any finite interval. We call $\mathcal{S} = \mathcal{S}(0)$ the subexponential class and $\mathcal{L} = \mathcal{L}(0)$ the class of long-tailed distributions. The intersection $\mathcal{S} \cap \mathcal{R}_{-\infty}$ contains a lot of well-known distributions such as the Weibull and the lognormal distributions. Typical examples in the classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$ with $\gamma > 0$ are the exponential distribution and the generalized inverse Gaussian distributions, respectively; see Embrechts (1983).

It is easy to verify the following statements, which will be tacitly used in the sequel.

1. for distributions F_1 and F_2, if $\overline{F}_1(x) \sim c\overline{F}_2(x)$ for some constant $c > 0$, then F_1 belongs to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ whenever F_2 belongs to this class (see Klüppelberg 1989, p. 260);

2. for any random variable X and any constant $c > 0$, if the distribution of X belongs to the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$, then the distribution of cX belongs to the class $\mathcal{S}(\gamma/c)$;

3. for two distributions F_1 and F_2, if $F_1 \in \mathcal{L}(\gamma_1)$ and $F_2 \in \mathcal{L}(\gamma_2)$ for some $0 \leq \gamma_1 < \gamma_2 < \infty$, then $\overline{F}_2(x) = o(\overline{F}_1(x))$.

Till now we have introduced four of the most important classes of heavy-tailed distributions. They are the classes $\mathcal{D}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{L}$. Another useful class is the so-called Extended Regular Variation (ERV) class. By definition, a distribution F belongs to the class ERV if there are some $0 \leq \alpha \leq \beta < \infty$ such that the relation

$$\theta^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} \leq \theta^{-\alpha}$$

holds for all $\theta > 1$. In this case we write $F \in \mathrm{ERV}(-\alpha, -\beta)$. This class has recently been applied to the study of precise large deviations; see Klüppelberg and Mikosch (1997), Ng et al. (2003), and references therein. It is well known that

$$\mathcal{R} \subset \mathrm{ERV} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}.$$

Actually, the inclusions $\mathcal{R} \subset \mathrm{ERV} \subset \mathcal{D} \cap \mathcal{L}$ can be verified directly by definition and the other inclusions $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ can be found in Embrechts et al. (1997, Chapters 1.3, 1.4, and A3) and references therein.

Let $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leq \alpha \leq \beta < \infty$. From Proposition 2.2.1 of Bingham et al. (1987) - see also Cline and Samorodnitsky (1994, Section 3) and Tang and Tsitsiashvili (2003, Section 3.3) - we know that, for any $p_1 < \alpha$ and $p_2 > \beta$, there are positive constants C_i and D_i, $i = 1, 2$, such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \geq C_1 \left(x/y\right)^{p_1} \tag{2.2}$$

holds whenever $x \geq y \geq D_1$, and that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C_2 \left(x/y\right)^{p_2} \tag{2.3}$$

holds whenever $x \geq y \geq D_2$. Furthermore, fixing the variable y in (2.3) leads to

$$x^{-p} = o\left(\overline{F}(x)\right) \qquad \text{for } p > \beta. \tag{2.4}$$

Hence, $\mathbb{E}\left(X^+\right)^p = \infty$ for all $p > \beta$.

For a distribution F and a real number x_0, denote by $F(x_0-)$ the right limit of F at $x = x_0$. The purpose of the assumption $F(0-)G(0-) = 0$ below is to guarantee that the equality

$$\overline{F \otimes G}(x) = \int_0^\infty \overline{F}(x/y)G(\mathrm{d}y)$$

holds for $x \geq 0$. The first of the following two lemmas is a reformulation of Theorem 2.1 of Cline and Samorodnitsky (1994).

Lemma 2.1. *Let F and G be two distributions with $F \in \mathcal{S}$, G nondegenerate at 0, and $F(0-)G(0-) = 0$. Then $H = F \otimes G \in \mathcal{S}$ if there is a positive function $a(\cdot)$ such that $a(x) = o(x)$, $\overline{F}(x - a(x)) \sim \overline{F}(x)$, and $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$.*

We establish a similar result for the class $\mathcal{R}_{-\infty}$ as follows.

Lemma 2.2. *Let F and G be two distributions with $F \in \mathcal{R}_{-\infty}$, G nondegenerate at 0, and $F(0-)G(0-) = 0$. Then $H = F \otimes G \in \mathcal{R}_{-\infty}$ if there is a positive function $a(\cdot)$ such that $a(x) = o(x)$ and $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$.*

Proof. For any $\theta > 1$ we have

$$\begin{aligned}
\frac{\overline{H}(\theta x)}{\overline{H}(x)} &= \frac{\left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \overline{F}\,(\theta x/y)\, G(\mathrm{d}y)}{\overline{H}(x)} \\
&\leq \frac{\int_0^{a(x)} \overline{F}\,(\theta x/y)\, G(\mathrm{d}y)}{\int_0^{a(x)} \overline{F}\,(x/y)\, G(\mathrm{d}y)} + \frac{\overline{G}(a(x))}{\overline{H}(x)} \\
&\leq \sup_{0<y\leq a(x)} \frac{\overline{F}\,(\theta x/y)}{\overline{F}\,(x/y)} + \frac{\overline{G}(a(x))}{\overline{H}(x)} \to 0.
\end{aligned}$$

Hence, $F \in \mathcal{R}_{-\infty}$. □

Trivially, if the distribution G only has a bounded support, the existence of the auxiliary function $a(\cdot)$ in Lemmas 2.1 and 2.2 is guaranteed by identifying it as a large constant. See also Corollary 2.5 of Cline and Samorodnitsky (1994) for a related discussion.

The following result is from Rogozin and Sgibnev (1999).

Lemma 2.3. *Let F, F_1, and F_2 be three distributions such that $F \in \mathcal{S}\,(\gamma)$ for $\gamma \geq 0$ and that the limit*

$$k_i = \lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}(x)}$$

exists and is finite for $i = 1, 2$. Then,

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = k_1 \int_{-\infty}^{\infty} \exp\{\gamma x\} F_2(\mathrm{d}x) + k_2 \int_{-\infty}^{\infty} \exp\{\gamma x\} F_1(\mathrm{d}x).$$

3 Uniform estimates with extended regular variation

Let us go back to the discrete time risk model introduced in Section 1. Hereafter, we always denote by F and G the distributions of the insurance risk X and the financial risk Y, respectively.

The following result, which originates from Theorem 5.1 of Tang and Tsitsiashvili (2003), establishes a uniform asymptotic relation for $\psi(x, n)$ with respect to $n = 1, 2, \ldots$.

Theorem 3.1. *If $F \in \mathrm{ERV}\,(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ and $\mathbb{E}\max\{Y^{\alpha-\delta}, Y^{\beta+\delta}\} < 1$ for some $0 < \delta < \alpha$, then the relation*

$$\psi(x, n) \sim \sum_{i=1}^{n} \mathbb{P}\left(X \prod_{j=1}^{i} Y_j > x\right) \tag{3.1}$$

holds uniformly for $n = 1, 2, \ldots$. *That is,*

$$\lim_{x\to\infty} \sup_{n\geq 1} \left| \frac{\psi(x,n)}{\sum_{i=1}^{n} \mathbb{P}\left(X \prod_{j=1}^{i} Y_j > x\right)} - 1 \right| = 0.$$

Proof. Clearly, the second condition above implies $\mathbb{E}Y^p < 1$ for any $p \in [\alpha - \delta, \beta + \delta]$. Choose some p_1 and p_2 satisfying

$$0 < \alpha - \delta < p_1 < \alpha \leq \beta < p_2 < \beta + \delta.$$

Then, there are positive constants C_i and D_i, $i = 1, 2$, such that inequalities (2.2) and (2.3) hold accordingly.

Write

$$\Delta_m = \sum_{i=m+1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j \qquad \text{for } m = 0, 1, \ldots.$$

We follow the proofs of Lemma 4.24 of Resnick (1987) and Proposition 1.1 of Davis and Resnick (1988) to show that $\mathbb{P}(\Delta_m > x)$ is asymptotically negligible when compared with $\overline{F}(x)$ in case x and m are sufficiently large. See also Embrechts et al. (1997, Section A3.3) for a simpler treatment. For all integers m such that $\sum_{i=m+1}^{\infty} i^{-2} < 1$, we derive that

$$\begin{aligned} \mathbb{P}(\Delta_m > x) &\leq \mathbb{P}\left(\sum_{i=m+1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j > \sum_{i=m+1}^{\infty} \frac{x}{i^2} \right) \\ &\leq \mathbb{P}\left(\bigcup_{i=m+1}^{\infty} \left(X_i^+ \prod_{j=1}^{i} Y_j > \frac{x}{i^2} \right) \right) \\ &\leq \sum_{i=m+1}^{\infty} \mathbb{P}\left(X_i^+ \prod_{j=1}^{i} Y_j > \frac{x}{i^2} \right). \end{aligned} \tag{3.2}$$

For all $i = 1, 2, \ldots$ and $x > 0$, introduce the events

$$\begin{aligned} A_1(i,x) &= \left(0 < i^{-2} \prod_{j=1}^{i} Y_j^{-1} \leq \frac{D_2}{x} \right), \\ A_2(i,x) &= \left(\frac{D_2}{x} < i^{-2} \prod_{j=1}^{i} Y_j^{-1} \leq 1 \right), \\ A_3(i,x) &= \left(1 < i^{-2} \prod_{j=1}^{i} Y_j^{-1} < \infty \right). \end{aligned}$$

We divide the right-hand side of (3.2) into three parts as $I_1(m,x)+I_2(m,x)+I_3(m,x)$ with

$$I_k(m,x) = \sum_{i=m+1}^{\infty} \mathbb{E}\left[\mathbb{P}\left(X_i > i^{-2}\prod_{j=1}^{i} Y_j^{-1}x \,\middle|\, Y_1,\ldots,Y_i\right) 1_{A_k(i,x)}\right],$$

where $1_{A_k(\cdot,\cdot)}$ denotes the indicator function of the event $A_k(\cdot,\cdot)$, $k=1,2,3$. Clearly,

$$I_1(m,x) \le \sum_{i=m+1}^{\infty} \mathbb{P}\left(A_1(i,x)\right) = \sum_{i=m+1}^{\infty} \mathbb{P}\left(i^2 \prod_{j=1}^{i} Y_j \ge \frac{x}{D_2}\right).$$

Since $\mathbb{E}Y^{\beta+\delta} < 1$ and relation (2.4) holds for $p=\beta+\delta$, applying Chebyshev's inequality we have

$$\lim_{m\to\infty}\limsup_{x\to\infty} \frac{I_1(m,x)}{\overline{F}(x)} = 0.$$

Applying inequalities (2.3) and (2.2) with p_2 and p_1 given above, for all $x \ge \max\{D_1, D_2\}$ we obtain, respectively,

$$I_2(m,x) \le C_2\overline{F}(x) \sum_{i=m+1}^{\infty} \mathbb{E}\left[i^{2p_2} \prod_{j=1}^{i} Y_j^{p_2} 1_{A_2(i,x)}\right] \le C_2\overline{F}(x) \sum_{i=m+1}^{\infty} i^{2p_2} \left(\mathbb{E}Y^{p_2}\right)^i,$$

and

$$I_3(m,x) \le \frac{\overline{F}(x)}{C_1} \sum_{i=m+1}^{\infty} \mathbb{E}\left[\left(i^{-2}\prod_{j=1}^{i} Y_j^{-1}\right)^{-p_1} 1_{A_3(i,x)}\right] \le \frac{\overline{F}(x)}{C_1} \sum_{i=m+1}^{\infty} i^{2p_1} \left(\mathbb{E}Y^{p_1}\right)^i.$$

Hence,

$$\lim_{m\to\infty}\limsup_{x\to\infty} \frac{I_k(m,x)}{\overline{F}(x)} = 0 \qquad \text{for } k=2,3.$$

Substituting these results into (3.2) leads to

$$\lim_{m\to\infty}\limsup_{x\to\infty} \frac{\mathbb{P}\left(\Delta_m > x\right)}{\overline{F}(x)} = \lim_{m\to\infty}\limsup_{x\to\infty} \sum_{i=m+1}^{\infty} \frac{\mathbb{P}\left(X_i^+ \prod_{j=1}^{i} Y_j > i^{-2}x\right)}{\overline{F}(x)} = 0.$$

Since $\mathbb{P}\left(XY > x\right) \asymp \overline{F}(x)$ - see Cline and Samorodnitsky (1994, Theorem 3.5(v)), it follows that

$$\lim_{m\to\infty}\limsup_{x\to\infty} \frac{\mathbb{P}\left(\Delta_m > x\right)}{\mathbb{P}\left(XY > x\right)} = \lim_{m\to\infty}\limsup_{x\to\infty} \sum_{i=m+1}^{\infty} \frac{\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right)}{\mathbb{P}\left(XY > x\right)} = 0. \tag{3.3}$$

By (3.3), for an arbitrarily fixed $0 < \varepsilon < 1$, there are some integer m_0 and some number $x_1 > 0$ such that

$$\mathbb{P}\left(\Delta_{m_0} > x\right) \le \varepsilon \mathbb{P}\left(XY > x\right) \tag{3.4}$$

and

$$\sum_{i=m_0+1}^{\infty} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \leq \varepsilon \mathbb{P}\left(XY > x\right) \tag{3.5}$$

hold for all $x \geq x_1$.

For the fixed m_0, applying Theorem 5.1 of Tang and Tsitsiashvili (2003), we have that relation (3.1) holds uniformly for $1 \leq n \leq m_0$. That is, the two-sided inequality

$$(1-\varepsilon) \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \leq \psi(x, n) \leq (1+\varepsilon) \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \tag{3.6}$$

holds for all $1 \leq n \leq m_0$ and $x \geq x_2$ for some $x_2 \geq x_1$.

Now we apply inequalities (3.4) and (3.5) to consider $n > m_0$. By (3.6) with $n = m_0$ and (3.5), it holds uniformly for $n > m_0$ and $x \geq x_2$ that

$$\begin{aligned} \psi(x, n) &\geq \psi(x, m_0) \\ &\geq (1-\varepsilon) \left(\sum_{i=1}^{n} - \sum_{i=m_0+1}^{n}\right) \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \\ &\geq (1-\varepsilon)^2 \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right). \end{aligned} \tag{3.7}$$

Next we aim at an upper bound for $\psi(x, n)$ with $n > m_0$. For any $0 < l < 1/2$,

$$\begin{aligned} \psi(x, n) &\leq \mathbb{P}\left(U_{m_0} + \sum_{i=m_0+1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\right) \\ &\leq \mathbb{P}\left(U_{m_0} + \Delta_{m_0} > x\right) \\ &\leq \mathbb{P}\left(U_{m_0} > (1-l)x\right) + \mathbb{P}\left(\Delta_{m_0} > lx\right) \\ &= J_1(l, x) + J_2(l, x). \end{aligned} \tag{3.8}$$

Applying inequality (3.6) with $n = m_0$, it holds for all $x \geq 2x_2$ that

$$J_1(l, x) \leq (1+\varepsilon) \sum_{i=1}^{m_0} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > (1-l)x\right).$$

Since $\mathbb{E}Y^{\beta+\delta} < 1$, by Theorem 3.5(iii) of Cline and Samorodnitsky (1994) we know that for any $i = 1, 2, \ldots$, the distribution of the product $X_i \prod_{j=1}^{i} Y_j$ still belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$. This means that

$$\sum_{i=1}^{m_0} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > (1-l)x\right) \lesssim (1-l)^{-\beta} \sum_{i=1}^{m_0} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right).$$

We identify the number l in (3.8) as some $l_0 > 0$ small enough such that $(1-l_0)^{-\beta} < 1+\varepsilon$. Therefore, for all $n > m_0$ and $x \geq x_3$ for some $x_3 \geq 2x_2$,

$$J_1(l_0, x) \leq (1+\varepsilon)^2 \sum_{i=1}^{m_0} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \leq (1+\varepsilon)^2 \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right). \tag{3.9}$$

As for $J_2(l_0, x)$, by (3.4) we have

$$J_2(l_0, x) \lesssim \varepsilon \mathbb{P}(XY > l_0 x) \lesssim \varepsilon l_0^{-\beta} \mathbb{P}(XY > x).$$

That is, for all $x \geq x_4$ for some $x_4 \geq x_3$,

$$J_2(l_0, x) \leq 2\varepsilon l_0^{-\beta} \mathbb{P}(XY > x). \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8) and recalling (3.7), we obtain that, uniformly for $n > m_0$ and $x \geq x_4$,

$$(1-\varepsilon)^2 \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \leq \psi(x, n) \leq \left((1+\varepsilon)^2 + 2\varepsilon l_0^{-\beta}\right) \sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right).$$

Combining this with (3.6) and taking into account the arbitrariness of $\varepsilon > 0$, we finally obtain the uniformity of relation (3.1) with respect to $n = 1, 2, \ldots$. □

The following is an immediate but important consequence of Theorem 3.1.

Theorem 3.2. *If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ and $\mathbb{E}\max\left\{Y^{\alpha-\delta}, Y^{\alpha+\delta}\right\} < 1$ for some $0 < \delta < \alpha$, then, uniformly for $n = 1, 2, \ldots$,*

$$\psi(x, n) \sim \frac{\mathbb{E}Y^\alpha \left(1 - (\mathbb{E}Y^\alpha)^n\right)}{1 - \mathbb{E}Y^\alpha} \overline{F}(x). \tag{3.11}$$

Proof. Using an elementary property - which is often referred to as Breiman's (1965) result - of the class $\mathcal{R}_{-\alpha}$, for each fixed $i = 1, 2, \ldots$,

$$\mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \sim (\mathbb{E}Y^\alpha)^i \overline{F}(x).$$

Hence, for each $n = 1, 2, \ldots$,

$$\sum_{i=1}^{n} \mathbb{P}\left(X_i \prod_{j=1}^{i} Y_j > x\right) \sim \frac{\mathbb{E}Y^\alpha \left(1 - (\mathbb{E}Y^\alpha)^n\right)}{1 - \mathbb{E}Y^\alpha} \overline{F}(x). \tag{3.12}$$

Similarly to the proof of Theorem 3.1, by inequality (3.5) it is easy to check that relation (3.12) holds uniformly for $n = 1, 2, \ldots$. Then applying Theorem 3.1, we immediately complete the proof of Theorem 3.2. □

Sometimes it is interesting to find asymptotic estimates for the ruin probability $\psi(x,n)$ in the case where both x and n tend to ∞ at a certain rate. This is the situation that is usually considered in large-deviation theory. Clearly, the uniformity of relation (3.11) enables us to derive such asymptotics. For example, we have the following result.

Corollary 3.1. *Let the conditions of Theorem 3.2 be valid. Then,*

1. *for any function* $n(x) : (0,\infty) \to \{1,2,\ldots\}$,
$$\psi(x,n(x)) \sim \frac{\mathbb{E}Y^\alpha \left(1-(\mathbb{E}Y^\alpha)^{n(x)}\right)}{1-\mathbb{E}Y^\alpha}\overline{F}(x);$$
2. *for any function* $x(n) : \{1,2,\ldots\} \to (0,\infty)$ *with* $\lim_{n\to\infty} x(n) = \infty$,
$$\psi(x(n),n) \sim \frac{\mathbb{E}Y^\alpha}{1-\mathbb{E}Y^\alpha}\overline{F}(x(n)), \qquad n \to \infty;$$
3. *it holds that*
$$\psi(x) \sim \frac{\mathbb{E}Y^\alpha}{1-\mathbb{E}Y^\alpha}\overline{F}(x). \tag{3.13}$$

Relation (3.13) gives a completely explicit estimate for the ultimate ruin probability in the presence of stochastic returns. The reader may compare this result with relation (1.8). The convergence rate of the ruin probability $\psi(x)$ given by (3.13) is not necessarily an exact power rate. The difference between the two results is not surprising as they are obtained under different conditions. In fact, recalling (1.5), one sees that the inequality $w > t_0$ holds under the conditions of Theorem 3.2.

4 Estimates with rapid variation

We denote by
$$y_* = y_*(G) = \sup\{y : G(y) < 1\}$$
the (upper) endpoint of the distribution G.

4.1 Case 1: $0 < y_*(G) \leq 1$

Since $Y_1 = (1+R_1)^{-1}$, the assumption $y_* \leq 1$ means that the insurer invests all his surplus into a risk-free asset and then he receives nonnegative stochastic returns. This case was not considered by Nyrhinen (1999, 2001) since the quantity w defined in (1.5) is infinite.

The infinite time ruin probability in continuous or discrete time models with a constant interest rate has been deeply investigated in the literature; see Sundt and Teugels (1995, 1997), Klüppelberg and Stadtmüller (1998), Asmussen (1998), Yang (1999), Kalashnikov and Konstantinides (2000), Konstantinides et al. (2002), and Tang (2004), among others.

Theorem 4.1. *Suppose that $F \in \mathcal{S}(\gamma)$ with $\gamma \geq 0$ and that G has an endpoint $y_* \leq 1$. Then, for each $n = 1, 2, \ldots$,*

$$\psi(x,n) \sim \sum_{i=1}^{n} \left(\mathbb{E}\exp\{\gamma X\}\right)^{i-1} \mathbb{E}\exp\{\gamma V_{n-i}\}\mathbb{P}\left(X\prod_{j=1}^{i} Y_j > x\right). \tag{4.1}$$

In particular, for $\gamma = 0$, formula (4.1) coincides with (3.1).

Proof. We derive from (1.9) and (1.10) that

$$\psi(x,1) = \mathbb{P}(V_1 > x) = \mathbb{P}(Y_1 X_1 > x).$$

Hence, (4.1) holds for $n = 1$. In addition, by Lemma A.4 we know that the distribution of V_1 belongs to the class $\mathcal{L}(\gamma/y_*)$.

Now we inductively assume that (4.1) holds for $n = m - 1$ for some integer $m \geq 2$ and that the distribution of V_{m-1} belongs to the class $\mathcal{L}(\gamma/y_*^{m-1})$. If $y_* = 1$ or $\gamma = 0$, noting that relation (4.1) with $n = m - 1$ implies $\mathbb{P}(V_{m-1} > x) = O\left(\overline{F}(x)\right)$, applying Lemma 3.2 of Tang and Tsitsiashvili (2003) we have

$$\mathbb{P}\left(X_m + V_{m-1} > x\right) \sim \mathbb{E}\exp\{\gamma V_{m-1}\}\overline{F}(x) + \mathbb{E}\exp\{\gamma X_m\}\mathbb{P}(V_{m-1} > x); \tag{4.2}$$

if $y_* < 1$ and $\gamma > 0$, then $\mathbb{P}(V_{m-1} > x) = o\left(\overline{F}(x)\right)$, and therefore, applying Lemma 2.3,

$$\mathbb{P}\left(X_m + V_{m-1} > x\right) \sim \mathbb{E}\exp\{\gamma X_m\}\mathbb{P}(V_{m-1} > x).$$

Thus, in any case, relation (4.2) holds. Successively applying (1.9), (1.10), and (4.2),

$$\begin{aligned}
\psi(x,m) &= \int_0^{y_*} \mathbb{P}(X_m + V_{m-1} > x/y)G(\mathrm{d}y) \\
&\sim \int_0^{y_*} \left(\mathbb{E}\exp\{\gamma V_{m-1}\}\overline{F}(x/y) + \mathbb{E}\exp\{\gamma X_m\}\mathbb{P}(V_{m-1} > x/y)\right) G(\mathrm{d}y) \\
&= \mathbb{E}\exp\{\gamma V_{m-1}\}\mathbb{P}(XY > x) + \mathbb{E}\exp\{\gamma X_m\}\int_0^{y_*} \psi(x/y, m-1)G(\mathrm{d}y).
\end{aligned}$$

Substituting to the above the asymptotic result (4.1) with $m - 1$ and x/y instead of n and x, after some simple adjustments we obtain (4.1) for $n = m$. This further indicates that the distribution of V_m belongs to the class $\mathcal{L}(\gamma/y_*^m)$.

The mathematical induction method completes the proof of Theorem 4.1. □

A natural consequence of Theorem 4.1 is the following, in which the assumption $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ with $\gamma \geq 0$ means that either $F \in \mathcal{S}(\gamma)$ with $\gamma > 0$ or $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ holds.

Corollary 4.1. *In addition to the conditions of Theorem 4.1 we assume $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ with $\gamma \geq 0$.*

1. *If* $\mathbb{P}(Y=1)=0$ *- the endpoint* y_* *may be less than or equal to* 1*, then,*

$$\psi(x,n) \sim \mathbb{E}\exp\{\gamma V_{n-1}\}\mathbb{P}\left(XY>x\right), \qquad n=1,2,\ldots. \tag{4.3}$$

In particular, for $\gamma=0$*, formula (4.3) can be simplified to*

$$\psi(x,n) \sim \mathbb{P}\left(XY>x\right), \qquad n=1,2,\ldots. \tag{4.4}$$

2. *If* $\mathbb{P}(Y=1)=p_*>0$*, then,*

$$\psi(x,n) \sim C_n(\gamma)\overline{F}(x), \qquad n=1,2,\ldots, \tag{4.5}$$

where

$$C_n(\gamma)=\sum_{i=1}^{n} p_*^i \left(\mathbb{E}\exp\{\gamma X\}\right)^{i-1}\mathbb{E}\exp\{\gamma V_{n-i}\}, \qquad n=1,2,\ldots.$$

In particular, for $\gamma=0$*, formula (4.5) can be simplified to*

$$\psi(x,n) \sim \left(p_*+p_*^2+\ldots+p_*^n\right)\overline{F}(x), \qquad n=1,2,\ldots.$$

Proof. In order to prove relation (4.3), it suffices to verify that the other terms on the right-hand side of (4.1) are asymptotically negligible when compared with $\mathbb{P}\left(XY>x\right)$. In fact, by Lemma 2.2 the distribution of the product XY belongs to the class $\mathcal{R}_{-\infty}$. For any $i\geq 2$, applying the dominated convergence theorem,

$$\lim_{x\to\infty}\frac{\mathbb{P}\left(X\prod_{j=1}^{i}Y_j>x\right)}{\mathbb{P}\left(XY>x\right)}=\int_{(0,1)}\lim_{x\to\infty}\frac{\mathbb{P}\left(XY>x/t\right)}{\mathbb{P}\left(XY>x\right)}G^{\otimes(i-1)}(\mathrm{d}t)=0.$$

Relation (4.5) can be proved similarly by applying (A.4) to (4.1). □

In case $\gamma>0$, the expressions for the coefficients in formulae (4.1), (4.3), and (4.5) are rather involved and it does not seem to admit a substantial simplification. However, the convergence rates characterized by these formulae are explicit.

Using a different approach Sgibnev (1996) proved (4.5) for the special case $p_*=1$.

In the following result we make the statement of relation (4.4) somewhat stronger.

Theorem 4.2. *Suppose that* $F\in\mathcal{S}\cap\mathcal{R}_{-\infty}$ *and that* G *has an endpoint* $0<y_*<1$*. Then it holds uniformly for* $n=1,2,\ldots$ *that*

$$\psi(x)\sim\psi(x,n)\sim\mathbb{P}\left(XY>x\right).$$

Proof. Trivially, it holds for all $n = 1, 2, \ldots$ and $x \geq 0$ that

$$\psi(x) \geq \psi(x, n) \geq \mathbb{P}\left(XY > x\right). \tag{4.6}$$

Hence, it suffices to establish appropriate upper bounds for $\psi(x)$ and $\psi(x, n)$. To this end, we notice that, for all $n, m = 1, 2, \ldots$ and $x \geq 0$,

$$\psi(x, n) \leq \psi(x) \leq \mathbb{P}\left(U_m + \sum_{i=m+1}^{\infty} y_*^i X_i^+ > x\right). \tag{4.7}$$

For an arbitrarily fixed number $\overline{y} \in (y_*, 1)$, we identify the integer m in (4.7) as some m_0 satisfying

$$\sum_{i=m_0+1}^{\infty} \overline{y}^i < 1 \qquad \text{and} \qquad \overline{p} = \mathbb{P}\left(Y > (y_*/\overline{y})^{m_0}\right) > 0.$$

For the first term in the last bracket of (4.7), by relation (4.4) we have

$$\mathbb{P}\left(U_{m_0} > x\right) = \psi(x, m_0) \sim \mathbb{P}\left(XY > x\right).$$

Hence by Lemma 2.1, the quantity U_{m_0} is subexponentially distributed. For the second term there, we derive that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=m_0+1}^{\infty} y_*^i X_i^+ > x\right) &\leq \mathbb{P}\left(\sum_{i=m_0+1}^{\infty} y_*^i X_i^+ > \sum_{i=m_0+1}^{\infty} \overline{y}^i x\right) \\ &\leq \mathbb{P}\left(\bigcup_{i=m_0+1}^{\infty} \left(y_*^i X_i^+ > \overline{y}^i x\right)\right) \\ &\leq \sum_{i=m_0+1}^{\infty} \mathbb{P}\left(X > \left(\overline{y}/y_*\right)^i x\right). \end{aligned}$$

Applying Fatou's lemma guaranteed by inequality (2.1) with $K > 1$, we obtain that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=m_0+1}^{\infty} y_*^i X_i^+ > x\right)}{\mathbb{P}\left(XY > x\right)} \leq \limsup_{x\to\infty} \frac{1}{\overline{p}} \sum_{i=m_0+1}^{\infty} \frac{\mathbb{P}\left(X > \left(\overline{y}/y_*\right)^i x\right)}{\mathbb{P}\left(X > \left(\overline{y}/y_*\right)^{m_0} x\right)} = 0.$$

By Lemma 2.3, it follows that

$$\mathbb{P}\left(U_{m_0} + \sum_{i=m_0+1}^{\infty} y_*^i X_i^+ > x\right) \sim \mathbb{P}\left(XY > x\right).$$

Substituting this into (4.7), we obtain that, uniformly for $n = 1, 2, \ldots$,

$$\psi(x, n) \leq \psi(x) \lesssim \mathbb{P}\left(XY > x\right). \tag{4.8}$$

By inequalities (4.6) and (4.8), we complete the proof of Theorem 4.2. $\square$

4.2 Case 2: $1 < y_*(G) < \infty$

Now we consider a more realistic case where negative investment returns may be earned.

Theorem 4.3. *Suppose that $F \in \mathcal{R}_{-\infty}$ and that G has an endpoint $1 < y_* < \infty$ with $p_* = \mathbb{P}(Y = y_*) > 0$. Then, for each $n = 1, 2, \ldots,$*

$$\psi(x,n) \sim p_*^n \mathbb{P}\left(\sum_{i=1}^{n} y_*^i X_i > x\right). \tag{4.9}$$

Proof. We derive from (1.9), (1.10), and (A.4) that

$$\psi(x,1) \sim p_* \mathbb{P}(y_* X_1 > x).$$

This implies that relation (4.9) holds for $n = 1$, that V_1 is rapidly varying tailed, and that $\overline{F}(lx) = o\left(\mathbb{P}\left(V_1 > x\right)\right)$ for $l = (y_*^{-1} + 1)/2 \in (y_*^{-1}, 1)$.

Now we inductively assume that relation (4.9) holds for $n = m - 1$ for some integer $m \geq 2$, that V_{m-1} is rapidly varying tailed, and that $\overline{F}(lx) = o\left(\mathbb{P}\left(V_{m-1} > x\right)\right)$. Hence by Lemma A.6,

$$\mathbb{P}\left(X_m + V_{m-1} > x\right) \sim p_*^{m-1} \mathbb{P}\left(X_m + \sum_{i=1}^{m-1} y_*^i X_i > x\right); \tag{4.10}$$

furthermore, by Lemma A.2 the sum $X_m + V_{m-1}$ is rapidly varying tailed. By (1.9), (1.10), and Lemma A.3 once again, we have

$$\psi(x,m) \sim p_* \mathbb{P}\left(y_*\left(X_m + V_{m-1}\right) > x\right). \tag{4.11}$$

Substituting (4.10) into (4.11) and noticing that X_n, $n = 1, 2, \ldots$ are i.i.d., we obtain (4.9) with $n = m$. This also proves that V_m is rapidly varying tailed. Moreover,

$$\frac{\overline{F}(lx)}{\mathbb{P}\left(V_m > x\right)} \sim \frac{1}{p_*^m} \frac{\overline{F}(lx)}{\mathbb{P}\left(\sum\limits_{i=1}^{m} y_*^i X_i > x\right)} \leq \frac{1}{p_*^m \left(\overline{F}(0)\right)^{m-1}} \frac{\overline{F}(lx)}{\overline{F}(x/y_*)} \to 0.$$

The mathematical induction method completes the proof of Theorem 4.3. □

If we restrict ourselves to the case $F \in \mathcal{S}(\gamma)$ with $\gamma \geq 0$, a completely explicit result can be derived.

Corollary 4.2. *Suppose that $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ with $\gamma \geq 0$ and that G has an endpoint $1 < y_* < \infty$ with $p_* = \mathbb{P}(Y = y_*) > 0$. Then, for each $n = 1, 2, \ldots,$*

$$\psi(x,n) \sim p_*^n \overline{F}\left(x y_*^{-n}\right) \prod_{i=1}^{n-1} \mathbb{E} \exp\left\{\gamma y_*^{-i} X\right\}. \tag{4.12}$$

In particular, for $\gamma = 0$, formula (4.12) can be simplified to

$$\psi(x,n) \sim p_*^n \overline{F}\left(x y_*^{-n}\right) \qquad \text{for each } n = 1, 2, \ldots. \tag{4.13}$$

Proof. Observe the right-hand side of (4.9). The distribution of $y_*^n X_n$ in the bracket belongs to the class $\mathcal{S}(\gamma/y_*^n)$ with tail $\overline{F}(xy_*^{-n})$. Compared with $\overline{F}(xy_*^{-n})$, the tail probabilities of the other terms in the bracket are asymptotically negligible. Hence, applying Lemma 2.3 we obtain

$$\psi(x,n) \sim p_*^n \mathbb{P}\left(\sum_{i=1}^{n-1} y_*^i X_i + y_*^n X_n > x\right) \sim p_*^n \mathbb{E}\exp\left\{\frac{\gamma}{y_*^n}\sum_{i=1}^{n-1} y_*^i X_i\right\}\mathbb{P}\left(y_*^n X_n > x\right).$$

This proves relation (4.12). □

Formula (4.13) can also be derived from (4.14) below.

In this subsection we assumed that the financial risk Y has a finite endpoint $Y_* > 1$ with a positive mass. This assumption is reasonable if Y is modelled by a positive, discrete, and bounded random variable. Secondly, suppose that the underlying financial risk in the economic environment is $\overline{Y} \in (0,\infty)$. When the insurer invests his surplus into a risky asset he always buys an option to hedge the downside risks. The resulting financial risk is modified by this strategy as

$$Y = \overline{Y} 1_{\left(0<\overline{Y}<y_*\right)} + y_* 1_{\left(y_*\le\overline{Y}<\infty\right)}$$

for some $y_* > 1$. Thus, it has a positive mass $\mathbb{P}\left(y_* \le \overline{Y} < \infty\right)$ at its endpoint y_*. Finally, we provide the following example as the third explanation for the assumption $\mathbb{P}(Y = y_*) > 0$.

Example 4.1. *In practice, there is a regulation that the insurer can only invest a part of his surplus into a risky asset. If the default risk appears he loses all the money invested into the risky asset and the default probability is positive. Assume that the insurer invests* $a \in (0,1)$*, say, of his surplus into a risk-free asset, which produces a constant return rate* $r > 0$*, and he invests the remaining surplus into a risky asset, which leads to a stochastic return rate* $R \in [-1,\infty)$ *with* $\mathbb{P}(R = -1) = p_* > 0$*. Let* $\widetilde{R}$ *be the overall return rate. Then,*

$$\widetilde{R} = ar + (1-a)R.$$

Clearly, the financial risk Y*, which is described by*

$$Y = \left(1 + ar + (1-a)R\right)^{-1},$$

is bounded from above by $y_* = \left(a(1+r)\right)^{-1}$ *with a positive mass* $\mathbb{P}(Y = y_*) = p_*$*.*

4.3 Case 3: $1 < y_*(G) \le \infty$

In this subsection we will not care whether the endpoint of the distribution G is finite or whether G has a positive mass at its endpoint. For notational convenience, we denote by H_n the distribution of the product $X\prod_{j=1}^n Y_j$ for $n = 1, 2, \ldots$.

Theorem 4.4. *Suppose $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and $\overline{G}(1) = \mathbb{P}(Y > 1) > 0$. If there is a positive function $a(\cdot)$ such that $a(x)/x \to 0$ eventually monotonically, $\overline{F}(x - a(x)) \sim \overline{F}(x)$, and $\overline{G}(a(x)) = o\left(\overline{H}_1(x)\right)$, then, for each $n = 1, 2, \ldots$,*

$$\psi(x, n) \sim \mathbb{P}\left(X \prod_{j=1}^{n} Y_j > x\right). \tag{4.14}$$

Proof. Clearly, for each $n = 1, 2, \ldots$,

$$\limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{H}_n(x)} \leq \limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{H}_1(x)\overline{G}^{n-1}(1)} = 0. \tag{4.15}$$

From (1.9) and (1.10), it is trivial that (4.14) holds for $n = 1$. In addition, by Lemma 2.1 we know $H_1 \in \mathcal{S}$, and by Lemma 2.2 we also know $H_1 \in \mathcal{R}_{-\infty}$. Moreover, we can prove that

$$\overline{H}_1(x - a(x)) \sim \overline{H}_1(x).$$

Actually, since $\overline{F}(x - a(x)) \sim \overline{F}(x)$, $\overline{G}(1) > 0$, and $a(x)/y \lesssim a(x/y)$ for all $1 < y \leq a(x)$, by Lemma A.3 we have

$$\begin{aligned}
\overline{H}_1(x) &\leq \overline{H}_1(x - a(x)) \\
&\sim \left(\int_1^{a(x)} + \int_{a(x)}^{\infty}\right) \overline{F}\left(\frac{x - a(x)}{y}\right) G(\mathrm{d}y) \\
&\lesssim \int_1^{a(x)} \overline{F}\left(x/y - a(x/y)\right) G(\mathrm{d}y) + \overline{G}(a(x)) \\
&= (1 + o(1)) \int_1^{a(x)} \overline{F}\left(x/y\right) G(\mathrm{d}y) + o\left(\overline{H}_1(x)\right) \\
&\lesssim \overline{H}_1(x).
\end{aligned} \tag{4.16}$$

Next we inductively assume that (4.14) holds for $n = m - 1$ for some integer $m \geq 2$, that $H_{m-1} \in \mathcal{S} \cap \mathcal{R}_{-\infty}$, and that

$$\overline{H}_{m-1}(x - a(x)) \sim \overline{H}_{m-1}(x). \tag{4.17}$$

We aim to prove (4.14) for $n = m$. By the right continuity of the distribution G, the condition $\overline{G}(1) > 0$ implies that there is some $y_0 > 1$ such that $\overline{G}(y_0) > 0$. We obtain

$$\frac{\overline{F}(x)}{\mathbb{P}(V_{m-1} > x)} \sim \frac{\overline{F}(x)}{\overline{H}_{m-1}(x)} \leq \frac{\overline{F}(x)}{\overline{F}(x/y_0)\overline{G}^{m-2}(1)\overline{G}(y_0)} \to 0. \tag{4.18}$$

Hence by Lemma 2.3,

$$\mathbb{P}(X_m + V_{m-1} > x) \sim \mathbb{P}(V_{m-1} > x) \sim \overline{H}_{m-1}(x).$$

From this, (1.9), and (1.10), we derive that

$$\begin{aligned}\psi(x,m) &= \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \mathbb{P}\left(X_m + V_{m-1} > x/y\right) G(\mathrm{d}y) \\ &= (1+o(1)) \int_0^{a(x)} \overline{H}_{m-1}\left(x/y\right) G(\mathrm{d}y) + \int_{a(x)}^{\infty} \mathbb{P}\left(X_m + V_{m-1} > x/y\right) G(\mathrm{d}y) \\ &= (1+o(1))\overline{H}_m(x) + \Xi_m(x),\end{aligned}$$

where the symbol $\Xi_m(x)$ denotes

$$\Xi_m(x) = \int_{a(x)}^{\infty} \mathbb{P}\left(X_m + V_{m-1} > x/y\right) G(\mathrm{d}y) - (1+o(1)) \int_{a(x)}^{\infty} \overline{H}_{m-1}\left(x/y\right) G(\mathrm{d}y).$$

Clearly, from (4.15),

$$\limsup_{x\to\infty} \frac{|\Xi_m(x)|}{\overline{H}_m(x)} \le 2 \limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{H}_m(x)} = 0.$$

This proves that (4.14) holds for $n = m$. In order for the mathematical induction to be complete, we have to prove $H_m \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and

$$\overline{H}_m(x - a(x)) \sim \overline{H}_m(x). \tag{4.19}$$

Recalling (4.17) and the inductive assumption $H_{m-1} \in \mathcal{S}\cap\mathcal{R}_{-\infty}$, the proof of $H_m \in \mathcal{S}\cap\mathcal{R}_{-\infty}$ is a direct application of Lemmas 2.1 and 2.2; the proof of (4.19) can be given by using (4.17) and copying the proof of (4.16) with H_1 being changed into H_m and F being changed into H_{m-1}.

Finally, the mathematical induction method completes the proof of Theorem 4.4. □

From the proof of Theorem 4.4, we see that the distribution H_n belongs to the class $\mathcal{R}_{-\infty}$ for $n = 1, 2, \ldots$. Similar to (4.18), for each $n = 1, 2, \ldots$,

$$\frac{\overline{H}_{n-1}(x)}{\overline{H}_n(x)} \le \frac{\overline{H}_{n-1}(x)}{\overline{H}_{n-1}(x/y_0)\overline{G}(y_0)} \to 0.$$

This gives the following consequence of Theorem 4.4.

Corollary 4.3. *Under the conditions of Theorem 4.4, it holds for each $n = 1, 2, \ldots$ that*

$$\psi(x,n) \sim \sum_{i=1}^{n} \mathbb{P}\left(X \prod_{j=1}^{i} Y_j > x\right) \sim \mathbb{P}\left(X \prod_{j=1}^{n} Y_j > x\right). \tag{4.20}$$

When the financial risk Y follows a lognormal distribution, as that in the Black-Scholes model, the calculation of the estimates given by (4.20) becomes particularly simple. See below:

Example 4.2. *Let F be lognormal-like with a density function*

$$f(x) \sim \frac{1}{cx} \exp\left\{-\frac{(\ln x - \mu_1)^2}{2\sigma_1^2}\right\} = \widetilde{f}(x)$$

for some $c > 0$, $-\infty < \mu_1 < \infty$, $\sigma_1 > 0$, and let G be lognormal with a density function

$$g(x) = \frac{1}{\sigma_2\sqrt{2\pi}x} \exp\left\{-\frac{(\ln x - \mu_2)^2}{2\sigma_2^2}\right\}$$

for $-\infty < \mu_2 < \infty$ and $\sigma_2 > 0$. Suppose $\sigma_2 < \sigma_1$.

We choose the auxiliary function in Theorem 4.4 as $a(x) = x^\alpha$ for some $\alpha \in (\sigma_2/\sigma_1, 1)$. Then, it is straightforward to verify $\overline{F}(x-a(x)) \sim \overline{F}(x)$ and $\overline{G}(a(x)) = o\left(\overline{F}(x)\right)$. Hence, all the conditions of Theorem 4.4 are satisfied and relation (4.20) holds for each $n = 1, 2, \ldots$.

Furthermore, for this concrete case, it is well known that the product $\prod_{j=1}^{i} Y_j$ also has a lognormal distribution with a density function

$$g_i(x) = \frac{1}{i\sigma_2\sqrt{2\pi}x} \exp\left\{-\frac{(\ln x - i\mu_2)^2}{2i^2\sigma_2^2}\right\}, \qquad i = 1, 2, \ldots.$$

By relation (4.20) and Lemma A.5, one sees that for each $n = 1, 2, \ldots$,

$$\psi(x, n) \sim \sum_{i=1}^{n} \iint_{v>0, uv>x} \widetilde{f}(u) g_i(v) \mathrm{d}u\mathrm{d}v \sim \iint_{v>0, uv>x} \widetilde{f}(u) g_n(v) \mathrm{d}u\mathrm{d}v.$$

5 Appendix

In this section we establish some results that were applied in the paper.

Lemma A.1. *If $F_1 \in \mathcal{R}_{-\infty}$ and $\overline{F}_2(x) = O\left(\overline{F}_1(x)\right)$, then $F = F_1 * F_2 \in \mathcal{R}_{-\infty}$.*

Proof. We formulate the proof into two parts according to whether or not $y_* = y_*(F_2)$, the endpoint of F_2, is finite.

First we assume $y_* < \infty$. Let $y_0 < y_*$ be a constant. Clearly, for any $x > 0$,

$$\overline{F}_1(x - y_*) \geq \overline{F}(x) \geq \int_{y_0}^{y_*} \overline{F}_1(x - y) F_2(\mathrm{d}y) \geq F_2(y_0, y_*] \overline{F}_1(x - y_0),$$

where $F_2(y_0, y_*] = F_2(y_*) - F_2(y_0)$. It follows that, for any $\theta > 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} \leq \frac{1}{F_2(y_0, y_*]} \limsup_{x\to\infty} \frac{\overline{F}_1(\theta x - y_*)}{\overline{F}_1(x - y_0)} = 0.$$

This proves $F \in \mathcal{R}_{-\infty}$.

Next we assume $y_* = \infty$. Then, for any θ and l with $\theta > 1$ and $1/\theta < l < 1$,

$$\begin{aligned}
\frac{\overline{F}(\theta x)}{\overline{F}(x)} &= \frac{\left(\int_{-\infty}^{\theta l x} + \int_{\theta l x}^{\infty}\right) \overline{F}_1(\theta x - y) F_2(\mathrm{d}y)}{\overline{F}(x)} \\
&\le \frac{\int_{-\infty}^{lx} \frac{\overline{F}_1(\theta x - \theta z)}{\overline{F}_1(x-z)} \overline{F}_1(x-z) F_2(\theta \mathrm{d}z) + \overline{F}_2(\theta l x)}{\overline{F}(x)} \\
&\le \sup_{-\infty < z \le lx} \frac{\overline{F}_1(\theta x - \theta z)}{\overline{F}_1(x-z)} \cdot \frac{\int_{-\infty}^{\theta l x} \overline{F}_1(x - \frac{y}{\theta}) F_2(\mathrm{d}y)}{\overline{F}(x)} + \frac{\overline{F}_2(\theta l x)}{\overline{F}(x)} \\
&= I_1 I_2 + I_3. \qquad \text{(A.1)}
\end{aligned}$$

Clearly, $I_1 \to 0$. We also have

$$I_3 \le \frac{\overline{F}_2(\theta l x)}{\overline{F}_1(\theta l x)} \frac{\overline{F}_1(\theta l x)}{\overline{F}_1(x)} \frac{1}{\overline{F}_2(0)} \to 0. \qquad \text{(A.2)}$$

As for I_2, it holds that

$$I_2 \le \frac{\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \overline{F}_1(x - \frac{y}{\theta}) F_2(\mathrm{d}y)}{\int_0^\infty \overline{F}_1(x-y) F_2(\mathrm{d}y)} \le \frac{\overline{F}_1(x) F_2(0)}{\overline{F}_1(x) \overline{F}_2(0)} + 1 = \frac{1}{\overline{F}_2(0)}.$$

Substituting these results into (A.1) yields that $\overline{F}(\theta x)/\overline{F}(x) \to 0$. □

Lemma A.2. *If $F_i \in \mathcal{R}_{-\infty}$ for $i = 1, 2$, then $F_1 * F_2 \in \mathcal{R}_{-\infty}$.*

Proof. The result can be obtained by copying the proof of Lemma A.1, with a modification on (A.2) in the following way:

$$I_3 \le \frac{\overline{F}_2(\theta l x)}{\overline{F}_2(x) \overline{F}_1(0)} \to 0.$$

This completes the proof. □

Lemma A.3. *Let X and Y be two independent random variables distributed by F and G, respectively. If $F \in \mathcal{R}_{-\infty}$, G has an endpoint $0 < y_* \le \infty$, and $F(0-)G(0-) = 0$, then for any $0 \le y_0 < y_*$,*

$$\mathbb{P}(XY > x) \sim \int_{y_0}^{y_*} \mathbb{P}(yX > x) G(\mathrm{d}y), \qquad \text{(A.3)}$$

where the integral interval $(y_0, y_*]$ is understood as (y_0, ∞) in case $y_* = \infty$. If further $y_* < \infty$ and $p_* = \mathbb{P}(Y = y_*) > 0$, then

$$\mathbb{P}(XY > x) \sim p_* \mathbb{P}(y_* X > x). \qquad \text{(A.4)}$$

Proof. To prove the first assertion, for any $x > 0$ we write

$$\mathbb{P}(XY > x) = \left(\int_0^{y_0} + \int_{y_0}^{y_*} \right) \mathbb{P}(yX > x) G(\mathrm{d}y) = J_1 + J_2. \tag{A.5}$$

Arbitrarily choose $\overline{y} \in (y_0, y_*)$. An appeal to Fatou's lemma gives that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{J_1}{J_2} &\leq \frac{1}{G(\overline{y}, y_*]} \limsup_{x\to\infty} \frac{\int_0^{y_0} \mathbb{P}(yX > x) G(\mathrm{d}y)}{\mathbb{P}(\overline{y}X > x)} \\
&\leq \frac{1}{G(\overline{y}, y_*]} \int_0^{y_0} \limsup_{x\to\infty} \frac{\mathbb{P}(yX > x)}{\mathbb{P}(\overline{y}X > x)} G(\mathrm{d}y) = 0.
\end{aligned}$$

Substituting this into (A.5) yields (A.3).

The second assertion can be proved similarly. Actually, we write

$$\mathbb{P}(XY > x) = \int_{(0,y_*)} \mathbb{P}(yX > x) G(\mathrm{d}y) + p_* \mathbb{P}(y_* X > x) = J_3 + J_4.$$

The dominated convergence theorem proves $J_3 = o(J_4)$. Hence, (A.4) holds. □

Lemma A.4. *Let F and G be two distributions such that $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$, $F(0-)G(0-) = 0$, and $y_* = y_*(G) \in (0, \infty)$. Then, $H = F \otimes G \in \mathcal{L}(\gamma/y_*)$.*

Proof. If $\gamma = 0$, then the result is a consequence of Theorem 2.2(iii) of Cline and Samorodnitsky (1994). Thus, we only consider $\gamma > 0$. In this case $F \in \mathcal{R}_{-\infty}$. For any $t > 0$ and any $0 < y_0 < y_*$, applying Lemma A.3 twice, we have

$$\begin{aligned}
\overline{H}(x - t) &\sim \int_{y_0}^{y_*} \overline{F}((x - t)/y) G(\mathrm{d}y) \\
&\leq \int_{y_0}^{y_*} \overline{F}(x/y - t/y_0) G(\mathrm{d}y) \\
&\sim \exp\{\gamma t/y_0\} \int_{y_0}^{y_*} \overline{F}(x/y) G(\mathrm{d}y) \\
&\sim \exp\{\gamma t/y_0\} \overline{H}(x).
\end{aligned}$$

In a symmetrical way, we obtain that

$$\overline{H}(x - t) \gtrsim \exp\{\gamma t/y_*\} \overline{H}(x).$$

Hence, by the arbitrariness of $0 < y_0 < y_*$ we obtain that

$$\overline{H}(x - t) \sim \exp\{\gamma t/y_*\} \overline{H}(x),$$

which implies $H \in \mathcal{L}(\gamma/y_*)$. □

Lemma A.5. *Let F_1, F_2, and G be three distributions such that $F_i(0-)G(0-) = 0$ for $i = 1, 2$, $G \in \mathcal{R}_{-\infty}$, and $\overline{F}_1(x) \sim c\overline{F}_2(x)$ for some $0 < c < \infty$. Then,*

$$\overline{F_1 \otimes G}(x) \sim c\overline{F_2 \otimes G}(x). \tag{A.6}$$

Proof. From the condition $\overline{F}_1(x) \sim c\overline{F}_2(x)$ we know that, for any $0 < \varepsilon < c$ and all large x, say $x \geq y_0$ for some $y_0 > 0$,

$$(c - \varepsilon)\overline{F}_2(x) \leq \overline{F}_1(x) \leq (c + \varepsilon)\overline{F}_2(x). \tag{A.7}$$

Since $\overline{F}_i(y_0) > 0$ for $i = 1, 2$, by virtue of Lemma A.3, for $i = 1, 2$,

$$\overline{F_i \otimes G}(x) \sim \int_{y_0}^{\infty} \overline{G}\,(x/y)\, F_i(\mathrm{d}y) = \overline{G}\,(x/y_0)\, \overline{F}_i(y_0) + \int_0^{x/y_0} \overline{F}_i(x/y)G(\mathrm{d}y).$$

Substituting (A.7) to the above leads to

$$(c - \varepsilon)\overline{F_2 \otimes G}(x) \lesssim \overline{F_1 \otimes G}(x) \lesssim (c + \varepsilon)\overline{F_2 \otimes G}(x).$$

Hence, relation (A.6) follows from the arbitrariness of $0 < \varepsilon < c$. □

Lemma A.6. *Let F, F_1, and F_2 be three distributions such that $F_1 \in \mathcal{R}_{-\infty}$, $\overline{F}_1(x) \sim c\overline{F}_2(x)$ for some $0 < c < \infty$, and $\overline{F}(lx) = o(\overline{F}_1(x))$ for some $0 < l < 1$. Then*

$$\overline{F * F_1}(x) \sim c\overline{F * F_2}(x). \tag{A.8}$$

Proof. With some $l^* \in (l, 1)$, we derive

$$\begin{aligned} \overline{F * F_1}(x) &= \left(\int_{-\infty}^{l^*x} + \int_{l^*x}^{\infty}\right) \overline{F}_1(x - u)F(\mathrm{d}u) \\ &= (c + o(1)) \int_{-\infty}^{l^*x} \overline{F}_2(x - u)F(\mathrm{d}u) + \int_{l^*x}^{\infty} \overline{F}_1(x - u)F(\mathrm{d}u) \\ &= (c + o(1))\overline{F * F_2}(x) + \Xi(x), \end{aligned} \tag{A.9}$$

where

$$\Xi(x) = \int_{l^*x}^{\infty} \overline{F}_1(x - u)F(\mathrm{d}u) - (c + o(1)) \int_{l^*x}^{\infty} \overline{F}_2(x - u)F(\mathrm{d}u).$$

Choose $M > 0$ such that $\overline{F}(-M) > 0$. Clearly,

$$\frac{|\Xi(x)|}{\overline{F * F_2}(x)} \leq \frac{(c + 1 + |o(1)|)\overline{F}(l^*x)}{\overline{F}(-M)\overline{F}_2(x + M)} \to 0.$$

Substituting this into (A.9) yields (A.8). □

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