Weighted Sums of Subexponential Random Variables and Their Maxima

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Abstract

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of independent random variables with common subexponential distribution F, and let $\{w_k, k = 1, 2, \ldots\}$ be a sequence of positive numbers. Under some mild summability conditions, we establish simple asymptotic estimates for the extreme tail probabilities of the weighted sum $\sum_{k=1}^{n} w_k X_k$ and of the maxima of weighted sums $\max_{1\le m\le n} \sum_{k=1}^{m} w_k X_k$, subject to the requirement that they should hold uniformly for $n = 1, 2, \ldots$. A direct application of the result is to risk analysis, where the ruin probability is to be evaluated for a company having the gross loss X_k during the kth year with a discount or inflation factor w_k.

Keywords: asymptotics, Matuszewska indices, weighted sum, maxima, subexponentiality, tail probabilities, uniformity, ruin probability, discounted loss.

1 Introduction

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $F = 1 - \overline{F}$, and let $\{w_k, k = 1, 2, \ldots\}$

be a sequence of positive numbers such that the series

$$S_\infty(w) = \sum_{k=1}^{\infty} w_k X_k \tag{1.1}$$

is well defined. We are interested in the tail behavior of the series $S_\infty(w)$ and the maximum

$$M_{(\infty)}(w) = \sup_{m\geq 1} \sum_{k=1}^{m} w_k X_k. \tag{1.2}$$

The distribution of $S_\infty(w)$ is of interest because the marginal distribution of any stationary linear process

$$\sum_{k=-\infty}^{\infty} w_k X_{m-k}, \quad m = 0, \pm 1, \pm 2, \ldots,$$

for two-sided sequences $\{w_k, k = 0, \pm 1, \ldots\}$ and $\{X_k, k = 0, \pm 1, \ldots\}$ can be represented as the distribution of some $S_\infty(w)$ of form (1.1). We can also interpret model (1.1) as the discounted value of all future losses of a company if we regard the random variable X_k as the gross loss during the kth year and regard the coefficient w_k as the discount or inflation factor from year k to the present. In this way, the ultimate maximum $M_{(\infty)}(w)$ can be interpreted as the maximal discounted value of future losses and the tail probability $\mathbb{P}\left(M_{(\infty)}(w) > x\right)$ can be interpreted as the ultimate ruin probability with initial surplus $x \geq 0$.

Zerner (2002) investigated the integrability of series (1.1), showing how the integrability of $f\left(|S_\infty(w)|\right)$ for some positive increasing function f corresponds to the integrability of $g\left(|X_1|\right)$ for some other function g.

We investigate the subtle tail behavior of the quantities (1.1) and (1.2). More precisely, assuming that the distribution F is subexponential (see below for the definition) with a certain restriction, our target is to seek for a condition on the sequence $\{w_k, k = 1, 2, \ldots\}$ so that the asymptotic result

$$\mathbb{P}\left(M_{(\infty)}(w) > x\right) \sim \mathbb{P}\left(S_\infty(w) > x\right) \sim \sum_{k=1}^{\infty} \overline{F}(x/w_k) \tag{1.3}$$

holds as $x \to \infty$. Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise; for two positive functions $A(\cdot)$ and $B(\cdot)$, we write $A(x) \lesssim B(x)$ if $\limsup A(x)/B(x) \leq 1$, write $A(x) \gtrsim B(x)$ if $\liminf A(x)/B(x) \geq 1$, and write $A(x) \sim B(x)$ if both. The desired condition is

$$\sum_{k=1}^{\infty} w_k^{\delta} < \infty$$

for a relevant constant $0 < \delta < 1$; see condition (3.1) below. Specifically, this condition is fulfilled when $w_k = (1+r)^{-k}$ for $k = 1, 2, \ldots$ and some $r > 0$. This is the case recently considered by Tang (2004) in the actuarial science literature.

For closely related works, we refer the reader to Cline (1983), Resnick (1987), and Davis and Resnick (1988), among others. See also Embrechts et al. (1997, Chapter A3.3) for a short review.

The rest of this paper is organized as follows: Section 2 recalls some preliminaries about subexponential distributions, Section 3 presents the main result and its corollaries, and Section 4 proves the main result after preparing some lemmas.

2 Heavy-tailed Distributions

Like many recent researchers in the fields of applied probability and risk theory, we restrict our interest to the case of heavy-tailed random variables. A random variable X or its distribution F satisfying $\overline{F}(x) > 0$ for any $x \in (-\infty, \infty)$ is said to be heavy tailed to the right if $\mathbb{E} \exp\{\gamma X\} = \infty$ for any $\gamma > 0$.

One of the most important classes of heavy-tailed distributions is the subexponential class $\mathcal{S}$. By definition, a distribution F on $[0, \infty)$ is subexponential, denoted by $F \in \mathcal{S}$, if

$$\lim_{x\to\infty} \frac{\overline{F^{(n)}}(x)}{\overline{F}(x)} = n, \tag{2.1}$$

for some $n \geq 2$ (or, equivalently, for any $n \geq 2$), where $F^{(n)}$ denotes the n-fold convolution of F. More generally, a distribution F on $(-\infty, \infty)$ is still said to be subexponential if the distribution $F^{+}(x) = F(x)\mathbf{1}_{(0 \leq x < \infty)}$ is subexponential. It is easy to check that (2.1) still holds for the latter general case. Clearly, for a sequence of i.i.d. random variables $\{X_k, k = 1, 2, \ldots\}$ with common distribution $F \in \mathcal{S}$, from (2.1) it holds for each $n \geq 2$ that

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k > x\right) \sim \mathbb{P}\left(\max_{1\leq k\leq n} X_k > x\right). \tag{2.2}$$

Relation (2.2) explains why the class $\mathcal{S}$ can be used to model the sizes of large loss variables.

Closely related are the class $\mathcal{L}$ of long-tailed distributions and the class $\mathcal{D}$ of distributions with dominatedly varying tails. By definition, a distribution F on $(-\infty, \infty)$ belongs to the class $\mathcal{L}$ if the relation

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \tag{2.3}$$

for any $y > 0$ (or, equivalently, for some $y > 0$); F belongs to the class $\mathcal{D}$ if the relation

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < \infty \tag{2.4}$$

for any $0 < v < 1$ (or, equivalently, for some $0 < v < 1$). It is well known that

$$\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L};$$

see Embrechts et al. (1997, Chapters 1.3, 1.4, and A3) and references therein. A famous subclass of the intersection $\mathcal{D} \cap \mathcal{L}$ is $\mathcal{R}$, which is the class of distributions with regularly varying tails. By definition, a distribution F on $(-\infty, \infty)$ belongs to the class $\mathcal{R}$ if there is some $\alpha \geq 0$ such that the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

holds for any $y > 0$. Denote $F \in \mathcal{R}_{-\alpha}$ in this case. Note that the class $\mathcal{R}$ is the union of all $\mathcal{R}_{-\alpha}$ over the range $0 \leq \alpha < \infty$. A slightly larger subclass of the intersection $\mathcal{D} \cap \mathcal{L}$ is the so-called Extended Regularly Varying (ERV) class. By definition, a distribution F on $(-\infty, \infty)$ belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leq \alpha \leq \beta < \infty$ if the relation

$$y^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha}$$

holds for any $y > 1$. Thus $\mathcal{R}_{-\alpha} = \mathrm{ERV}(-\alpha, -\alpha)$. Note that the class ERV is the union of all $\mathrm{ERV}(-\alpha, -\beta)$ over the range $0 \leq \alpha \leq \beta < \infty$.

In this paper we will also consider another distribution class below, which complements the class $\mathcal{R}$ with an extreme case that the index α is equal to ∞.

Definition 2.1. *Let F be a distribution on $(-\infty, \infty)$. F is said to be rapidly varying tailed, denoted by $F \in \mathcal{R}_{-\infty}$, if*

$$\lim_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 0 \qquad \text{for each } v > 1. \tag{2.5}$$

We remark that a more general notion, rapid variation, has been extensively investigated in the literature; for details we refer the reader to the monographs de Haan (1970, Chapter 1.2) and Bingham et al. (1987, Chapter 2.4), among others.

We recall two significant indices, which are crucial for our purpose. Let F be a distribution on $(-\infty, \infty)$. As done recently by Tang and Tsitsiashvili (2003), for any $y > 0$ we set

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \qquad \overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \tag{2.6}$$

and then define

$$\mathbb{J}_F^+ = \inf\left\{ -\frac{\log \overline{F}_*(y)}{\log y} : y > 1 \right\}, \qquad \mathbb{J}_F^- = \sup\left\{ -\frac{\log \overline{F}^*(y)}{\log y} : y > 1 \right\}. \tag{2.7}$$

Following Tang and Tsitsiashvili (2003), we call the quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ above the upper and lower Matuszewska indices of the distribution F. Clearly, if $F \in \mathcal{R}_{-\alpha}$ for some $0 \le \alpha \le \infty$ then $\mathbb{J}_F^\pm = \alpha$, and if $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \le \alpha \le \beta < \infty$ then $\alpha \le \mathbb{J}_F^- \le \mathbb{J}_F^+ \le \beta$. For more details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1) and Cline and Samorodnitsky (1994).

Let F be a distribution with a lower Matuszewska index $0 < \mathbb{J}_F^- \le \infty$. With some simple adjustments on the second inequality of Proposition 2.2.1 of Bingham et al. (1987), we see that for any $0 < p < \mathbb{J}_F^-$, there are positive constants C_1 and x_0 such that the inequality

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \le C_1 y^{-p} \tag{2.8}$$

holds uniformly for $xy \ge x \ge x_0$. Fixing the variable $x = x_0$ in (2.8), we obtain that the inequality

$$\frac{\overline{F}(x_0 y)}{\overline{F}(x_0)} \le C_1 y^{-p}$$

holds uniformly for $y \ge 1$. Then, substituting $x = x_0 y$ to the above yields that for some constant $C_2 > 0$, the inequality

$$\overline{F}(x) \le C_2 x^{-p} \tag{2.9}$$

holds uniformly for $x \ge x_0$.

Recently, Konstantinides et al. (2002) introduced the following subclass of subexponential distributions:

Definition 2.2. *Let F be a distribution on $(-\infty, \infty)$. We say that F belongs to the class $\mathcal{A}$ if F is subexponential and has a lower Matuszewska index $0 < \mathbb{J}_F^- \le \infty$.*

Clearly, $0 < \mathbb{J}_F^- \le \infty$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < 1 \qquad \text{for some } y > 1.$$

The purpose to introduce this class is mainly to exclude some very heavy-tailed (like slowly varying tailed) distributions from the class $\mathcal{S}$. It is easy to see that

$$\mathcal{R}_{-\alpha} \subset \mathcal{A} \quad \text{for any } 0 < \alpha < \infty, \qquad \mathcal{S} \cap \mathcal{R}_{-\infty} \subset \mathcal{A}.$$

3 The Main Result and Its Corollaries

Hereafter, for two positive bivariate functions $A(x; n)$ and $B(x; n)$, we say that the asymptotic relation $A(x; n) \sim B(x; n)$ holds uniformly for $n = 1, 2, \ldots$ if

$$\lim_{x\to\infty} \sup_{n\ge1} \left| \frac{A(x; n)}{B(x; n)} - 1 \right| = 0;$$

that is, for any $\varepsilon > 0$ there is some $x(\varepsilon) > 0$ such that the two-sided inequality

$$(1-\varepsilon)B(x;n) \leq A(x;n) \leq (1+\varepsilon)B(x;n)$$

holds for all $x \geq x(\varepsilon)$ and all $n = 1, 2, \ldots$. Clearly, the asymptotic relation $A(x;n) \sim B(x;n)$ holds uniformly for $n = 1, 2, \ldots$ if and only if

$$\limsup_{x\to\infty} \sup_{n\geq 1} \frac{A(x;n)}{B(x;n)} \leq 1$$

and

$$\liminf_{x\to\infty} \inf_{n\geq 1} \frac{A(x;n)}{B(x;n)} \geq 1.$$

The latter two relations mean that $A(x;n) \lesssim B(x;n)$ and $A(x;n) \gtrsim B(x;n)$, respectively, hold uniformly for $n = 1, 2, \ldots$.

Recall that $\{X_k, k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with common distribution function F and that $\{w_k, k = 1, 2, \ldots\}$ is a sequence of positive numbers. For notational convenience, we write, for each $n = 1, 2, \ldots, \infty$,

$$S_n(w) = \sum_{k=1}^{n} w_k X_k, \quad M_{(n)}(w) = \max_{1\leq m\leq n} S_m(w), \quad M_n(w) = \max_{1\leq k\leq n} w_k X_k;$$

for a real number x, we write $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$; we then write, for each $n = 1, 2, \ldots, \infty$,

$$S_n^+(w) = \sum_{k=1}^{n} w_k X_k^+.$$

The main contribution of this paper is the following

Theorem 3.1. *Suppose that $F \in \mathcal{A}$ and*

$$\sum_{k=1}^{\infty} w_k^{\delta} < \infty \qquad \text{for some } 0 < \delta < \frac{\mathbb{J}_F^-}{1+\mathbb{J}_F^-}, \tag{3.1}$$

where $\mathbb{J}_F^-/(1+\mathbb{J}_F^-) = 1$ when $\mathbb{J}_F^- = \infty$. Then it holds uniformly for $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(M_{(n)}(w) > x\right) \sim \mathbb{P}\left(S_n^+(w) > x\right) \sim \mathbb{P}\left(M_n(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \tag{3.2}$$

If, in addition,

$$S_\infty^-(w) = \sum_{k=1}^{\infty} w_k X_k^- <_{a.s.} \infty, \tag{3.3}$$

then result (3.2) holds with $S_n^+(w)$ replaced by $S_n(w)$.

We suggest the reader to compare our result (3.2) with relation (2.2).

Now we verify the a.s. convergence of the series and maxima involved in (3.2). With convention $w_0 = 0$, we observe that

$$0 \le \max_{0 \le m < \infty} S_m(w) \le S_\infty^+(w).$$

Under condition (3.1), applying the well-known monotone convergence theorem and the C_r- inequality, we have

$$\mathbb{E}\left(S_\infty^+(w)\right)^\delta \le \mathbb{E}\left(X_1^+\right)^\delta \sum_{k=1}^\infty w_k^\delta < \infty,$$

where the finiteness of $\mathbb{E}\left(X_1^+\right)^\delta$ is guaranteed by inequality (2.9) since $\delta < \mathbb{J}_F^-$. This implies that the series $S_\infty^+(w)$ and the maximum $\max_{0 \le m < \infty} S_m(w)$ are a.s. finite; hence, the tail probabilities $\mathbb{P}\left(M_{(\infty)}(w) > x\right)$ and $\mathbb{P}\left(S_\infty^+(w) > x\right)$ are not reduced to trivial constants for $x > 0$. Similar explanations can be given for the tail probability $\mathbb{P}\left(M_\infty(w) > x\right)$. In addition, the convergence of the series $\sum_{k=1}^\infty \mathbb{P}\left(w_k X_k > x\right)$ for each $x > 0$ can also be verified since by inequality (2.8), it holds for some $\delta < p < \mathbb{J}_F^-$ and all large k and x that

$$\mathbb{P}\left(w_k X_k > x\right) \le C_1 w_k^p \overline{F}(x). \tag{3.4}$$

A recent large deviation result of Korshunov (2001) has a similar taste as Theorem 3.1 above. For a subexponential distribution F with $\int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$, let

$$\overline{F_u}(x) = \begin{cases} \min\left\{1, \int_x^{x+u} \overline{F}(y)\mathrm{d}y\right\}, & x \ge 0, \\ 1, & x < 0. \end{cases} \tag{3.5}$$

Clearly, for any $u \in [0, \infty]$, F_u defines a standard distribution on $[0, \infty)$. According to the terminology of Korshunov (2001), we say that the distribution F is strongly subexponential if the relation

$$\overline{F_u^{(2)}}(x) \sim 2\overline{F_u}(x)$$

holds uniformly for $u \in [1, \infty]$. Korshunov's result states that if the common distribution F of the increments of a random walk $S_n = \sum_{k=1}^n X_k$ is strongly subexponential and has a finite mean $-\mu = \mathbb{E}X_1 < 0$, then uniformly for $n = 1, 2, \ldots$,

$$\mathbb{P}\left(\max_{1 \le k \le n} S_k > x\right) \sim \frac{1}{\mu}\int_x^{x+n\mu} \overline{F}(y)\mathrm{d}y \sim \sum_{k=1}^n \overline{F}(x + k\mu). \tag{3.6}$$

Compared with Korshunov's result, our Theorem 3.1 successfully establishes a corresponding uniform asymptotic result for the case where the weights are included.

When each X_k has a distribution F_k satisfying $\overline{F_k}(x) \sim c_k \overline{F}(x)$ for some distribution $F \in \mathcal{S}$ and positive constants c_k, $k = 1, 2, \ldots$, Ng et al. (2002, Theorem 2.2) established a result similar to (3.2), but without weights and only for a fixed integer $n = 1, 2, \ldots$.

Clearly, the uniformity of the asymptotics in (3.2) allows for $n = \infty$. Hence, we have

Corollary 3.1. *Write* $P_1(x) = \mathbb{P}\left(M_{(\infty)}(w) > x\right)$, $P_2(x) = \mathbb{P}\left(S_\infty^+(w) > x\right)$, *and* $P_3(x) = \mathbb{P}\left(M_\infty(w) > x\right)$ *for* $x \geq 0$. *Under the conditions of Theorem 3.1, it holds that*

$$P_1(x) \sim P_2(x) \sim P_3(x) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(w_k X_k > x\right). \tag{3.7}$$

Specifically, if we choose $w_k = (1+r)^{-k}$, $k = 1, 2, \ldots$, with some $r > 0$, then the relation corresponding to the probability $P_1(x)$, which can be interpreted as the ultimate ruin probability of a discrete time model with constant interest rate $r > 0$, coincides with the main result of Tang (2004).

Now we put forward some special cases for Corollary 3.1 (the same thing can be done for Theorem 3.1).

Corollary 3.2. *Let* $P_i(x)$, $i = 1, 2, 3$, *be defined in Corollary 3.1.*

(1) Suppose that $F \in \mathcal{S}$ *with* $0 < \mathbb{J}_F^- \leq \mathbb{J}_F^+ < \infty$ *(or, equivalently,* $F \in \mathcal{D} \cap \mathcal{A}$*) and that condition (3.1) holds. Then for* $i = 1, 2, 3$,

$$0 < \sum_{k=1}^{\infty} \overline{F}_*(w_k^{-1}) \leq \liminf_{x \to \infty} \frac{P_i(x)}{\overline{F}(x)} \leq \limsup_{x \to \infty} \frac{P_i(x)}{\overline{F}(x)} \leq \sum_{k=1}^{\infty} \overline{F}^*(w_k^{-1}) < \infty. \tag{3.8}$$

(2) Suppose that $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \leq \beta < \infty$ *and that*

$$\sum_{k=1}^{\infty} w_k^{\delta} < \infty \qquad \text{for some } 0 < \delta < \frac{\alpha}{1+\alpha}. \tag{3.9}$$

Then for $i = 1, 2, 3$,

$$\overline{F}(x) \sum_{k=1}^{\infty} \min\left\{w_k^{\alpha}, w_k^{\beta}\right\} \lesssim P_i(x) \lesssim \overline{F}(x) \sum_{k=1}^{\infty} \max\left\{w_k^{\alpha}, w_k^{\beta}\right\}.$$

(3) Suppose that $F \in \mathcal{R}_{-\alpha}$ *for some* $0 < \alpha < \infty$ *and that condition (3.9) holds. Then for* $i = 1, 2, 3$,

$$P_i(x) \sim \overline{F}(x) \sum_{k=1}^{\infty} w_k^{\alpha}.$$

(4) Suppose that $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and that

$$\sum_{k=1}^{\infty} w_k^{\delta} < \infty \qquad \textit{for some } 0 < \delta < 1. \tag{3.10}$$

Then for $i = 1, 2, 3$,

$$P_i(x) \sim \overline{F}\left(x/w^*\right) \sum_{k=1}^{\infty} \mathbf{1}_{(w_k = w^*)}$$

with $w^* = \max\{w_k,\ k = 1, 2, \ldots\}$.

Proof. Clearly, item (3) is a consequence of item (2). Also, item (2) is a consequence of item (1) since $\alpha \leq \mathbb{J}_F^- \leq \mathbb{J}_F^+ \leq \beta$ for $F \in \mathrm{ERV}(-\alpha, -\beta)$. In order to prove item (1) we observe the right-hand side of relation (3.7). By inequality (2.8) with $p = \delta < \mathbb{J}_F^-$, inequality (3.4) holds for all large $k = 1, 2, \ldots$ and all large $x > 0$. Hence, applying Fatou's lemma justified by condition (3.1), we obtain

$$\limsup_{x \to \infty} \frac{P_i(x)}{\overline{F}(x)} \leq \sum_{k=1}^{\infty} \limsup_{x \to \infty} \frac{\mathbb{P}\left(w_k X_k > x\right)}{\overline{F}(x)} = \sum_{k=1}^{\infty} \overline{F}^*(w_k^{-1}).$$

The corresponding lower bound in (3.8) can be proved similarly.

Now we prove item (4). The convergence of the series $\sum_{k=1}^{\infty} \mathbf{1}_{(w_k = w^*)}$ is implied by condition (3.1). We divide the right-hand side of (3.7) into two parts as

$$\sum_{k=1}^{\infty} \mathbb{P}\left(w_k X_k > x\right) = \left(\sum_{k:\ w_k = w^*} + \sum_{k:\ w_k < w^*}\right) \mathbb{P}\left(w_k X_k > x\right) = \sum\nolimits_1 + \sum\nolimits_2 .$$

Recalling Definition 2.1, similarly to the proof of item (1), applying Fatou's lemma we have

$$\limsup_{x \to \infty} \frac{\sum_2}{\mathbb{P}\left(w^* X_1 > x\right)} \leq \sum_{k:\ w_k < w^*} \limsup_{x \to \infty} \frac{\mathbb{P}\left(w_k X_k > x\right)}{\mathbb{P}\left(w^* X_1 > x\right)} = 0.$$

So

$$\sum_{k=1}^{\infty} \mathbb{P}\left(w_k X_k > x\right) \sim \sum\nolimits_1 = \overline{F}\left(x/w^*\right) \sum_{k=1}^{\infty} \mathbf{1}_{(w_k = w^*)}.$$

This ends the proof of Corollary 3.2. $\square$

Works closely related to Corollaries 3.1 and 3.2 can be found in Cline (1983), Davis and Resnick (1988), as well as Embrechts et al. (1997, Chapter A3.3). Among them, for the case where the tail probability $\mathbb{P}\left(|X_k| > x\right)$ is regularly varying at infinity and the weights $\{w_k, k = 1, 2, \ldots\}$ are not necessarily positive, Cline (1983) established a result similar to Corollary 3.2(3) but without $P_1(x)$. For the case where the common distribution F is both in the domain of attraction of $\Lambda(x) = \exp\{-e^{-x}\}$ for $x \in (-\infty, \infty)$ and in the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, Davis and Resnick (1988) established a result similar to Corollary 3.2(4). Note that the distribution F being in the domain of attraction of $\Lambda(x)$ indicates $F \in \mathcal{R}_{-\infty}$.

4 Proof of the Main Result

4.1 Lemmas

Lemma 4.1. *Consider the convolution of two distributions F_1 and F_2 on $(-\infty, \infty)$. If $F_1 \in \mathcal{S}$ and $\overline{F_2}(x) \lesssim c\overline{F_1}(x)$ for some $c \geq 0$, then*

$$\overline{F_1 * F_2}(x) \lesssim (1 + c)\overline{F_1}(x). \tag{4.1}$$

Proof. See Lemma 4.4 of Tang (2004). □

Lemma 4.2. *Let $\{X_k, k = 1, 2, \ldots, n\}$ be n independent and real-valued random variables. If each X_k has a distribution $F_k \in \mathcal{L}$, $k = 1, 2, \ldots, n$, then*

$$\mathbb{P}\left(\max_{1\leq m\leq n} \sum_{k=1}^{m} X_k > x\right) \sim \mathbb{P}\left(\sum_{k=1}^{n} X_k > x\right). \tag{4.2}$$

Proof. See Theorem 2.1 of Ng et al. (2002). □

Lemma 4.3. *Let $\{X_k, k = 1, 2, \ldots, n\}$ be n i.i.d. and real-valued random variables with common distribution $F \in \mathcal{S}$, and let $\{w_k, k = 1, 2, \ldots, n\}$ be n positive numbers. Then, the weighted sum $S_n(w)$ is subexponentially distributed and satisfies*

$$\mathbb{P}\left(S_n(w) > x\right) \sim \sum_{k=1}^{n} \overline{F}(x/w_k). \tag{4.3}$$

Proof. We need a result from Tang and Tsitsiashvili (2003); see also Embrechts and Goldie (1980) and Cline (1986, Corollary 1). This result states that if $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$, and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F_1 * F_2 \in \mathcal{S}$ and

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x).$$

By this result one proves (4.3) for $n = 2$. Then it extends by induction. This ends the proof of Lemma 4.3. □

Lemma 4.4. *Under the conditions of Theorem 3.1, it holds that*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}\left(\sum_{k=n}^{\infty} w_k X_k^+ > x\right)}{\mathbb{P}\left(w_1 X_1 > x\right)} = 0 \tag{4.4}$$

and that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\sum_{k=n}^{\infty} \mathbb{P}\left(w_k X_k > x\right)}{\mathbb{P}\left(w_1 X_1 > x\right)} = 0. \tag{4.5}$$

Proof. We follow the proofs of Lemma 4.24 of Resnick (1987) and Proposition 1.1 of Davis and Resnick (1988); see also Embrechts et al. (1997, Chapter A3.3) for a simpler treatment.

First, we choose some p_1 such that

$$\delta < p_1 < 1 - \frac{\delta}{\mathbb{J}_F^-}.$$

Hence by condition (3.1), it holds that $\sum_{k=1}^{\infty} w_k^{p_1} < \infty$. For all large n such that

$$\sum_{k=n}^{\infty} w_k^{p_1} < 1 \quad \text{and} \quad w_1 w_k^{p_1-1} > 1 \text{ for all } k \geq n,$$

we have

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=n}^{\infty} w_k X_k^+ > x\right) &\leq \mathbb{P}\left(\sum_{k=n}^{\infty} w_k X_k^+ > \sum_{k=n}^{\infty} w_k^{p_1} x\right) \\
&\leq \mathbb{P}\left(\bigcup_{k=n}^{\infty} \left(w_k X_k^+ > w_k^{p_1} x\right)\right) \\
&\leq \sum_{k=n}^{\infty} \mathbb{P}\left(w_1 X_k^+ > w_1 w_k^{p_1-1} x\right).
\end{aligned} \tag{4.6}$$

Next, we choose some p_2 such that

$$0 < \frac{\delta}{1-p_1} < p_2 < \mathbb{J}_F^-.$$

Applying inequality (2.8) with $p = p_2$ to the right-hand side of (4.6) yields that

$$\sum_{k=n}^{\infty} \mathbb{P}\left(w_1 X_k^+ > w_1 w_k^{p_1-1} x\right) \leq C_1 \sum_{k=n}^{\infty} \left(w_1 w_k^{p_1-1}\right)^{-p_2} \mathbb{P}\left(w_1 X_1 > x\right) \tag{4.7}$$

holds for all large $x > 0$. Since $(1-p_1)p_2 > \delta$, by condition (3.1),

$$\sum_{k=1}^{\infty} \left(w_1 w_k^{p_1-1}\right)^{-p_2} < \infty.$$

Hence, inequalities (4.6) and (4.7) give the first result (4.4).

Observe that

$$\sum_{k=n}^{\infty} \mathbb{P}\left(w_k X_k > x\right) \leq \sum_{k=n}^{\infty} \mathbb{P}\left(w_1 X_k^+ > w_1 w_k^{p_1-1} x\right)$$

holds for all $x > 0$ and all large n such that $0 < w_k < 1$ for $k \geq n$, relation (4.5) is a natural consequence of inequality (4.7) and the convergence of the series $\sum_{k=1}^{\infty} \left(w_1 w_k^{p_1-1}\right)^{-p_2}$. This ends the proof of Lemma 4.4. $\square$

4.2 Proof of Theorem 3.1

We formulate the proof into four parts as follows.

1. Relation $\mathbb{P}\left(M_{(n)}(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)$ holds uniformly for n.

By Lemma 4.4, for any $0 < \varepsilon < 1$ there is some $m = m(\varepsilon) = 1, 2, \ldots$ such that

$$\mathbb{P}\left(\sum_{k=n}^{\infty} w_k X_k^+ > x\right) \leq \varepsilon \mathbb{P}\left(w_1 X_1 > x\right) \tag{4.8}$$

and

$$\sum_{k=n}^{\infty} \mathbb{P}\left(w_k X_k > x\right) \leq \varepsilon \mathbb{P}\left(w_1 X_1 > x\right) \tag{4.9}$$

hold for all large x and all $n > m$. For this fixed m and each $1 \leq n \leq m$, successively applying Lemmas 4.2 and 4.3 we have

$$\mathbb{P}\left(M_{(n)}(w) > x\right) \sim \mathbb{P}\left(S_n(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \tag{4.10}$$

Hence, there is some $A = A(\varepsilon) > 0$ such that for all $1 \leq n \leq m$ and $x \geq A$,

$$(1-\varepsilon) \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) \leq \mathbb{P}\left(M_{(n)}(w) > x\right) \leq (1+\varepsilon) \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \tag{4.11}$$

By Lemma 4.3, relation (4.10) further indicates that the partial sum $S_m(w)$ and the maximum $M_{(m)}(w)$ are subexponentially distributed.

Now we consider $n > m$. By (4.11), for $x \geq A$ we have that

$$\begin{aligned}
\mathbb{P}\left(M_{(n)}(w) > x\right) &\geq \mathbb{P}\left(M_{(m)}(w) > x\right) \\
&\geq (1-\varepsilon) \sum_{k=1}^{m} \mathbb{P}\left(w_k X_k > x\right) \\
&= (1-\varepsilon) \left(\sum_{k=1}^{n} - \sum_{k=m+1}^{n}\right) \mathbb{P}\left(w_k X_k > x\right).
\end{aligned} \tag{4.12}$$

Since, by (4.9), for all $n > m$ and all large x,

$$\sum_{k=m+1}^{n} \mathbb{P}\left(w_k X_k > x\right) \leq \sum_{k=m+1}^{\infty} \mathbb{P}\left(w_k X_k > x\right) \leq \varepsilon \mathbb{P}\left(w_1 X_1 > x\right).$$

it holds for all $n > m$ and all large x that

$$\begin{aligned}
\mathbb{P}\left(M_{(n)}(w) > x\right) &\geq (1-\varepsilon) \left(\sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) - \varepsilon \mathbb{P}\left(w_1 X_1 > x\right)\right) \\
&\geq (1-\varepsilon)^2 \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right).
\end{aligned} \tag{4.13}$$

Symmetrically, we derive upper bounds for $\mathbb{P}\left(M_{(n)}(w) > x\right)$. For all $n > m$,

$$\begin{aligned}\mathbb{P}\left(M_{(n)}(w) > x\right) &\le \mathbb{P}\left(M_{(m)}(w) + \sum_{k=m+1}^{n} w_k X_k^+ > x\right)\\ &\le \mathbb{P}\left(M_{(m)}(w) + \sum_{k=m+1}^{\infty} w_k X_k^+ > x\right).\end{aligned}$$

Since, by inequality (4.8),

$$\mathbb{P}\left(\sum_{k=m+1}^{\infty} w_k X_k^+ > x\right) \le \varepsilon \mathbb{P}\left(M_{(m)}(w) > x\right),$$

applying Lemma 4.1 and inequalities (4.11), we obtain that, uniformly for $n > m$,

$$\begin{aligned}\mathbb{P}\left(M_{(n)}(w) > x\right) &\lesssim (1+\varepsilon)\mathbb{P}\left(\max_{1\le n\le m}\sum_{k=1}^{n} w_k X_k > x\right)\\ &\le (1+\varepsilon)^2 \sum_{k=1}^{m} \mathbb{P}\left(w_k X_k > x\right)\\ &\le (1+\varepsilon)^2 \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \qquad (4.14)\end{aligned}$$

Combining (4.13) and (4.14) gives that, uniformly for $n > m$,

$$(1-\varepsilon)^2 \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) \lesssim \mathbb{P}\left(M_{(n)}(w) > x\right) \lesssim (1+\varepsilon)^2 \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \qquad (4.15)$$

¿From (4.11) and (4.15), we conclude that the two-sided inequality (4.15) actually holds uniformly for $n = 1, 2, \ldots$. Hence, the uniformity of

$$\mathbb{P}\left(M_{(n)}(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) \qquad (4.16)$$

follows from the arbitrariness of $\varepsilon > 0$.

2. Relation $\mathbb{P}\left(S_n^+(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)$ holds uniformly for n.

Observe that in the first step we have proved the uniformity of relation (4.16) with respect to $n = 1, 2, \ldots$. Applying this fact to the sequence $\{X_k^+, k = 1, 2, \ldots\}$ yields the desired result immediately.

3. Relation $\mathbb{P}\left(M_n(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)$ holds uniformly for n.

Trivially, for all $n = 1, 2, \ldots$ and all $x > 0$,

$$\mathbb{P}\left(M_n(w) > x\right) \le \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right).$$

It remains to prove the other inequality for the relation. To this end, we shall apply an elementary inequality that for n general events E_1, ..., E_n,

$$\mathbb{P}\left(\bigcup_{k=1}^{n} E_k\right) \geq \sum_{k=1}^{n} \mathbb{P}\left(E_k\right) - \sum_{1\leq k\neq l\leq n} \mathbb{P}\left(E_k E_l\right).$$

By this inequality, we derive that for all $n = 1, 2, \ldots$ and all $x > 0$,

$$\begin{aligned}
\mathbb{P}\left(M_n(w) > x\right) &\geq \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) - \sum_{1\leq k\neq l\leq n} \mathbb{P}\left(w_k X_k > x, w_l X_l > x\right) \\
&\geq \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right) - \left(\sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)\right)^2 \\
&\geq \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)\left(1 - \sum_{k=1}^{\infty} \mathbb{P}\left(w_k X_k > x\right)\right).
\end{aligned}$$

Relation (4.5) trivially implies that $\sum_{k=1}^{\infty} \mathbb{P}\left(w_k X_k > x\right)$ tends to 0 as $x \to \infty$. Hence, we obtain the desired result that uniformly for $n = 1, 2, \ldots$,

$$\mathbb{P}\left(M_n(w) > x\right) \gtrsim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right).$$

4. Under (3.3), relation $\mathbb{P}\left(S_n(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)$ holds uniformly for n. Applying the result proved in step 2, we know that the relation

$$\mathbb{P}\left(S_n(w) > x\right) \leq \mathbb{P}\left(S_n^+(w) > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right)$$

holds uniformly for $n = 1, 2, \ldots$. Thus, it remains to prove the uniformity of the relation

$$\mathbb{P}\left(S_n(w) > x\right) \gtrsim \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right). \tag{4.17}$$

We go back to the proof of step 1. From (4.10) we know that the two sides of relation (4.17) are asymptotically equal to each other for $1 \leq n \leq m$. For $n > m$, we have

$$\mathbb{P}\left(S_n(w) > x\right) \geq \mathbb{P}\left(S_m(w) - \sum_{k=m+1}^{n} w_k X_k^- > x\right) \geq \mathbb{P}\left(S_m(w) - \sum_{k=m+1}^{\infty} w_k X_k^- > x\right).$$

Note that under assumption (3.3) the series $U_m^-(w) = \sum_{k=m+1}^{\infty} w_k X_k^-$, say, is a well defined nonnegative random variable, that the random variables $S_m(w)$ and $U_m^-(w)$ are independent, and that the partial $S_m(w)$ is long tailed. Applying the dominated convergence theorem we obtain that

$$\mathbb{P}\left(S_m(w) - \sum_{k=m+1}^{\infty} w_k X_k^- > x\right) \sim \mathbb{P}\left(S_m(w) > x\right).$$

Hence for any $\varepsilon > 0$ and $n > m$, applying (4.10), similarly to the proof of step 1 we have

$$\begin{aligned}\mathbb{P}\left(S_n(w) > x\right) &\gtrsim \sum_{k=1}^{m} \mathbb{P}\left(w_k X_k > x\right) \\ &\geq \left(\sum_{k=1}^{n} - \sum_{k=m+1}^{\infty}\right) \mathbb{P}\left(w_k X_k > x\right) \\ &\gtrsim (1-\varepsilon) \sum_{k=1}^{n} \mathbb{P}\left(w_k X_k > x\right).\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ we conclude that relation (4.17) holds uniformly for $n = 1, 2, \ldots$. This ends the proof of Theorem 3.1. □

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