

# Sums of Dependent Nonnegative Random Variables with Subexponential Tails

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## Abstract

In this paper we study the asymptotic tail probabilities of sums of subexponential nonnegative random variables, which are dependent according to certain general structures with tail independence. The results show that the subexponentiality of the summands eliminates the impact of the dependence on the tail behavior of the sums.

*Keywords:* Asymptotics; copula; dependence; subexponentiality; uniformity.

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## 1 Introduction

We are interested in the tail probabilities of sums of dependent and heavy-tailed random variables. Throughout this paper, by saying that a distribution  $F$  is concentrated on  $[0, \infty)$  we mean that  $F(0-) = 0$  and  $\bar{F}(x) = 1 - F(x) > 0$  for all  $x > 0$ . One of the most important classes of heavy-tailed distributions is the subexponential class, written as  $\mathcal{S}$ . By definition, a distribution  $F$  concentrated on  $[0, \infty)$  belongs to the class  $\mathcal{S}$  if the relation

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{*n}}(x)}{\bar{F}(x)} = n \quad (1.1)$$

holds for some (or, equivalently, for all)  $n = 2, 3, \dots$ , where  $F^{*n}$  denotes the  $n$ -fold convolution of  $F$ . Hence, if  $X_1, X_2, \dots$  are independent and identically distributed (i.i.d.) random variables with common distribution  $F \in \mathcal{S}$ , then for every  $n = 2, 3, \dots$ ,

$$\Pr(S_n > x) \sim \Pr(X_{(n)} > x) \sim n\bar{F}(x). \quad (1.2)$$

Here and henceforth,  $S_n = X_1 + \dots + X_n$ ,  $X_{(n)} = \max\{X_1, \dots, X_n\}$ , all limit relationships are for  $x \rightarrow \infty$  unless stated otherwise, and  $a(x) \sim b(x)$  stands for  $\lim a(x)/b(x) = 1$ .

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We also need two other classes of heavy-tailed distributions. A distribution  $F$  is said to be long tailed, written as  $F \in \mathcal{L}$ , if the relation

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x-t)}{\overline{F}(x)} = 1$$

holds for some (or, equivalently, for all)  $t > 0$ , and  $F$  is said to be dominatedly varying tailed, written as  $F \in \mathcal{D}$ , if the relation

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(tx)}{\overline{F}(x)} < \infty$$

holds for some (or, equivalently, for all)  $0 < t < 1$ . It is well known that  $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ ; see, e.g. Section 1.4 of Embrechts *et al.* (1997).

Because the class  $\mathcal{S}$  enjoys the nice asymptotic property in (1.2), it has been extensively applied to modelling heavy-tailed distributions appearing in insurance and finance. However, owing to the very definition (1.1), the mainstream study of subexponentiality has been restricted to the i.i.d. case. This seriously limits the usefulness of the beautiful theory of subexponentiality in practice. The purpose of this paper is to find out to what extent  $X_1, \dots, X_n$  can be dependent while relations (1.2) remain, or, in other words, how insensitive the tail behavior of the sum  $S_n$  is on the dependence among its subexponential summands.

The insensitivity of subexponentiality has recently been observed by different researchers. For  $F \in \mathcal{D} \cap \mathcal{L}$ , Geluk and Ng (2006) proved relations (1.2) under the assumption that the random variables  $X_1, \dots, X_n$  are negatively associated, meaning that, for every pair of disjoint subsets  $I$  and  $J$  of  $\{1, \dots, n\}$ , the inequality  $\text{Cov}\{f(X_i, i \in I), g(X_j, j \in J)\} \leq 0$  holds for all coordinate-wise increasing functions  $f$  and  $g$  for which the covariance exists. Tang (2008) relaxed the dependence structure from negative association to pairwise negative quadrant dependence; see Remark 2.4, below, for the definition. Albrecher *et al.* (2006) studied the problem for  $n = 2$  using copulas to describe the dependence structure of  $X_1$  and  $X_2$ . Their Lemma 2.7 shows that relations (1.2) hold if  $F$  is absolutely continuous and the copula density of  $(X_1, X_2)$  exists and is uniformly bounded on  $[c, 1]^2$  for some  $0 < c < 1$ ; see also Remark 2.3, below, for details. Tang and Tsitsiashvili (2003) observed the insensitivity of subexponentiality from a different angle. Consider the weighted random variables  $X_1 = \omega_1 Y_1, \dots, X_n = \omega_n Y_n$ , where the primary random variables  $Y_1, \dots, Y_n$  are i.i.d. with common subexponential distribution while the random weights  $\omega_1, \dots, \omega_n$ , independent of the primary random variables, are dependent with common distribution concentrated on  $[a, b]$  for some  $0 < a < b < \infty$ . For this case, relations (1.2) still hold by Theorem 3.1 of Tang and Tsitsiashvili (2003). All these works reveal a phenomenon that, when deriving the tail asymptotics for the sum of dependent random variables, the subexponentiality assumption eliminates the impact of certain dependence structures among the summands.

In this paper we shall propose another general dependence structure, which does not require the existence of probability densities or copula densities and which allows both positive

and negative dependence to a certain extent. Our discussions in Section 2.1, below, show that our assumption is satisfied by many commonly used dependence structures.

The rest of this paper consists of two sections. Section 2 is devoted to the bivariate case under our basic assumption on the dependence between  $X_1$  and  $X_2$ , while Section 3 extends the scenario to the multivariate case under a modified version of the dependence assumption.

## 2 The Bivariate Case

### 2.1 Our Assumption on the Dependence

In this section we study the case in which  $n = 2$  with  $X_1$  and  $X_2$  distributed by  $F_1$  and  $F_2$  concentrated on  $[0, \infty)$ , respectively. Our basic assumption is as follows.

For  $(i, j) = (1, 2)$  or  $(2, 1)$ , the relation

$$\frac{\Pr(X_j > x - t | X_i = t)}{\Pr(X_j > x - t)} = O(1) \quad \text{Assumption A1}$$

holds uniformly for all  $t \in [x_0, x]$  for some large  $x_0 > 0$ .

Here the uniformity means

$$\limsup_{x \rightarrow \infty} \sup_{x_0 \leq t \leq x} \frac{\Pr(X_j > x - t | X_i = t)}{\Pr(X_j > x - t)} < \infty. \quad (2.1)$$

When  $t$  is not a possible value of  $X_i$ , i.e.  $\Pr(X_i \in \Delta) = 0$  for some open interval  $\Delta$  containing  $t$ , the conditional probability in Assumption A1 is simply understood as 0. Since

$$\sup_{x - x_0 < t \leq x} \frac{\Pr(X_j > x - t | X_i = t)}{\Pr(X_j > x - t)} \leq \frac{1}{F_j(x_0)},$$

relation (2.1) is equivalent to

$$\limsup_{x \rightarrow \infty} \sup_{x_0 \leq t \leq x - x_0} \frac{\Pr(X_j > x - t | X_i = t)}{\Pr(X_j > x - t)} < \infty. \quad (2.2)$$

Intuitively, Assumption A1 requires that the dependence structure between  $X_1$  and  $X_2$  should not be too positive. Hence, it excludes extremely positive dependence structures such as comonotonicity. However, Assumption A1 can still be satisfied by most of extremely negative dependence structures such as countermonotonicity. As an example, if  $X_1 X_2 = 1$  then relation (2.2) holds with  $x_0 = 1$ .

In the following remarks we show that Assumption A1 indeed allows a wide range of dependence structures. We only consider Assumption A1 with  $(i, j) = (1, 2)$ .

**Remark 2.1.** We show that Assumption A1 is satisfied if  $X_2$  is stochastically decreasing in  $X_1$ , written as  $\text{SD}(X_2 | X_1)$ , meaning that, for each fixed  $x_2$ , the conditional probability

$\Pr(X_2 > x_2 | X_1 = x_1)$  is decreasing (not necessarily strictly) in  $x_1$ . Actually, in this case, for every fixed  $x_0 \geq 0$  for which  $F_1(x_0) > 0$ , it holds uniformly for all  $t \geq x_0$  that

$$\begin{aligned} \Pr(X_2 > x - t | X_1 = t) &= \frac{\int_{0-}^{x_0} \Pr(X_2 > x - t | X_1 = t) dF_1(u)}{F_1(x_0)} \\ &\leq \frac{\int_{0-}^{x_0} \Pr(X_2 > x - t | X_1 = u) dF_1(u)}{F_1(x_0)} \leq \frac{\overline{F_2}(x - t)}{F_1(x_0)}. \end{aligned}$$

Hence, Assumption A1 is satisfied. We comment that  $\text{SD}(X_2 | X_1)$  reveals a flavor of negative dependence. In the literature this dependence structure is also called negative regression dependence; see, e.g. Lehmann (1966).  $\square$

**Remark 2.2.** For simplicity, let  $F_1$  and  $F_2$  be absolutely continuous and let  $X_1$  and  $X_2$  be dependent according to a copula  $C(u_1, u_2)$  for  $(u_1, u_2) \in [0, 1]^2$ . Thus, the joint distribution of  $X_1$  and  $X_2$  is given by

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2));$$

see, e.g. page 15 of Nelsen (2006). Let  $U_1 = F_1(X_1)$  and  $U_2 = F_2(X_2)$ , so that they are two uniform random variables following the joint distribution  $C(u_1, u_2)$ . Under Assumption A1, it is not difficult to verify that  $X_1$  and  $X_2$  are tail independent in the sense that the (upper) tail dependence measure, defined by

$$\chi = \lim_{u \rightarrow 1} \Pr(U_2 > u | U_1 > u),$$

is equal to 0. Indeed, for all  $u$  close to 1 such that  $F_1^{-1}(u) \geq x_0$  and  $F_2^{-1}(u) \geq x_0$ ,

$$\begin{aligned} \Pr(U_2 > u | U_1 > u) &= \int_{F_1^{-1}(u)}^{\infty} \frac{\Pr(X_2 > F_2^{-1}(u) | X_1 = t)}{1 - u} dF_1(t) \\ &\leq (1 - u) \sup_{F_1^{-1}(u) < t < \infty} \frac{\Pr(X_2 > F_2^{-1}(u) | X_1 = t)}{1 - u} \\ &\leq (1 - u) \sup_{x_0 \leq t \leq F_2^{-1}(u) + t} \frac{\Pr(X_2 > F_2^{-1}(u) + t - t | X_1 = t)}{\Pr(X_2 > F_2^{-1}(u) + t - t)}. \end{aligned}$$

Using the change of variables  $\tilde{x} = F_2^{-1}(u) + t$ , we find that, as  $u \rightarrow 1$ ,

$$\Pr(U_2 > u | U_1 > u) \leq (1 - u) \sup_{x_0 \leq t \leq \tilde{x}} \frac{\Pr(X_2 > \tilde{x} - t | X_1 = t)}{\Pr(X_2 > \tilde{x} - t)} = O(1 - u). \quad (2.3)$$

Hence,  $\chi = 0$  as claimed.

To overcome some limitations of the tail dependence measure  $\chi$ , Coles *et al.* (1999) defined another tail dependence measure as

$$\bar{\chi} = \lim_{u \rightarrow 1} \frac{2 \log \Pr(U_1 > u)}{\log \Pr(U_1 > u, U_2 > u)} - 1.$$

By (2.3), it is clear that  $\bar{\chi} \leq 0$  if it exists.  $\square$

**Remark 2.3.** Now we examine Assumption A1 from the perspective of copulas. Following the notation of Remark 2.2, further assume that the first-order partial derivative

$$C_1(u_1, u_2) = \frac{\partial}{\partial u_1} C(u_1, u_2)$$

exists. Then, by the copula representation of the conditional distribution, Assumption A1 with  $(i, j) = (1, 2)$  can be restated as that the relation

$$\frac{1 - C_1(F_1(t), F_2(x - t))}{F_2(x - t)} = O(1)$$

holds uniformly for all  $t \in [x_0, x]$  (or, equivalently, for all  $t \in [x_0, x - x_0]$ ) for some large  $x_0 > 0$ . Furthermore, if the second-order mixed partial derivative, usually called the copula density,

$$C_{12}(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2),$$

exists and is uniformly bounded by some constant  $M > 0$  for all  $(u_1, u_2) \in [c, 1]^2$  for some  $0 < c < 1$ , then Assumption A1 is satisfied. To see this, let us choose some  $x_0 > 0$  such that  $F_1(x_0) \geq c$  and  $F_2(x_0) \geq c$ . It holds, for all  $t \in [x_0, x - x_0]$ , that

$$\frac{\Pr(X_2 > x - t | X_1 = t)}{\Pr(X_2 > x - t)} = \frac{\int_{x-t}^{\infty} C_{12}(F_1(t), F_2(z)) dF_2(z)}{\Pr(X_2 > x - t)} \leq M.$$

Hence, relation (2.2) holds. This boundedness assumption on the copula density  $C_{12}(u_1, u_2)$  was proposed by Albrecher *et al.* (2006, Lemma 2.7). Restricting to Archimedean copulas, the verification of Assumption A1 becomes even easier by looking at the generator and its first two derivatives, as done in Albrecher *et al.* (2006, Lemma 3.5).  $\square$

**Remark 2.4.** Recall that random variables  $X_1$  and  $X_2$  are called positively quadrant dependent if

$$\Pr(X_1 \leq x_1, X_2 \leq x_2) \geq \Pr(X_1 \leq x_1) \Pr(X_2 \leq x_2),$$

or, equivalently,

$$\Pr(X_1 > x_1, X_2 > x_2) \geq \Pr(X_1 > x_1) \Pr(X_2 > x_2),$$

for all  $x_1, x_2$ . They are called negatively quadrant dependent if both inequalities above are in the reverse direction. Assumption A1 allows both positive and negative quadrant dependence structures to a certain extent. As an example, consider the Ali-Mikhail-Haq copula

$$C(u_1, u_2; \theta) = \frac{u_1 u_2}{1 - \theta(1 - u_1)(1 - u_2)}, \quad \theta \in [-1, 1),$$

which is a kind of Archimedean copula with generator  $\varphi(z; \theta) = \ln(1 - \theta(1 - z))/z$ . This copula can generate both positive and negative quadrant dependence structures depending on the sign of  $\theta$ . It is easy to verify that Assumption A1 is satisfied by this copula.  $\square$

## 2.2 The First Main Result and Its Proof

Now we state our first main result.

**Theorem 2.1.** *Let  $X_1$  and  $X_2$  be two random variables with distributions  $F_1$  and  $F_2$  concentrated on  $[0, \infty)$ , respectively, such that Assumption A1 holds. Then, the relations*

$$\Pr(S_2 > x) \sim \Pr(X_{(2)} > x) \sim \overline{F_1}(x) + \overline{F_2}(x) \quad (2.4)$$

hold for each of the following two cases:

$$1^\circ F_1 \in \mathcal{S}, F_2 \in \mathcal{L}, \text{ and } \overline{F_2}(x) = O(\overline{F_1}(x));$$

$$2^\circ F_i \in \mathcal{D} \cap \mathcal{L} \text{ for } i = 1, 2.$$

A by-product of this theorem is that the distribution of  $S_2$  belongs to  $\mathcal{S}$  for case 1 $^\circ$  and belongs to  $\mathcal{D} \cap \mathcal{L}$  for case 2 $^\circ$ . The key ingredient of our proof of Theorem 2.1 is to bound the probability of  $(X_1, X_2)$  falling into a trapezoid in the first quadrant above the line  $x_1 + x_2 = x$ . This is motivated by the proof of Lemma 2.7 of Albrecher *et al.* (2006).

Trivially, the relation  $a(x) \sim b(x)$  amounts to the conjunction of  $\limsup a(x)/b(x) \leq 1$  and  $\liminf a(x)/b(x) \geq 1$ , which are denoted by  $a(x) \lesssim b(x)$  and  $a(x) \gtrsim b(x)$ , respectively. By this and the fact  $S_2 \geq X_{(2)}$ , we see that relations (2.4) follow immediately from Lemmas 2.1 and 2.2, below.

**Lemma 2.1.** *Under the conditions of Theorem 2.1, it holds that*

$$\Pr(S_2 > x) \lesssim \overline{F_1}(x) + \overline{F_2}(x). \quad (2.5)$$

*Proof.* We prove Lemma 2.1 under Assumption A1 only with  $(i, j) = (1, 2)$ , and we remark that the proof for  $(i, j) = (2, 1)$  is the same. For arbitrarily fixed  $l \geq x_0$ , we have

$$\Pr(S_2 > x) \leq \Pr(X_1 > x) + \Pr(X_2 \geq x - l) + \Pr(S_2 > x, l < X_1 \leq x). \quad (2.6)$$

Since  $F_2$  is long tailed,  $\Pr(X_2 \geq x - l) \sim \overline{F_2}(x)$ . By Assumption A1, there exists some  $0 < C < \infty$ , which does not depend on  $l$  or  $x$ , such that, for all  $x \geq l \geq x_0$ ,

$$\sup_{l < t \leq x} \frac{\Pr(X_2 > x - t | X_1 = t)}{\overline{F_2}(x - t)} \leq C.$$

It follows that

$$\begin{aligned} \Pr(S_2 > x, l < X_1 \leq x) &= \int_l^x \Pr(X_2 > x - t | X_1 = t) dF_1(t) \\ &\leq C \int_l^x \overline{F_2}(x - t) dF_1(t) \\ &= C \left[ \int_{0-}^{\infty} \overline{F_2}(x - t) dF_1(t) - \int_{0-}^l \overline{F_2}(x - t) dF_1(t) - \overline{F_1}(x) \right]. \end{aligned}$$

The first integral in the bracket is asymptotically equal to  $\overline{F}_1(x) + \overline{F}_2(x)$ ; for case 1°, see Corollary 1 of Cline (1986) while, for case 2°, see Theorem 2.1 of Cai and Tang (2004). Hence, for every  $0 < \varepsilon < 1$  and for all large  $x$ ,

$$\begin{aligned} \Pr(S_2 > x, l < X_1 \leq x) &\leq C [(1 + \varepsilon) (\overline{F}_1(x) + \overline{F}_2(x)) - (1 - \varepsilon) F_1(l) \overline{F}_2(x) - \overline{F}_1(x)] \\ &= C [\varepsilon \overline{F}_1(x) + (\overline{F}_1(l) + \varepsilon + \varepsilon F_1(l)) \overline{F}_2(x)]. \end{aligned}$$

Substituting all these estimates into (2.6) yields that, for all large  $x$ ,

$$\begin{aligned} \Pr(S_2 > x) &\leq \overline{F}_1(x) + (1 + \varepsilon) \overline{F}_2(x) + C [\varepsilon \overline{F}_1(x) + (\overline{F}_1(l) + \varepsilon + \varepsilon F_1(l)) \overline{F}_2(x)] \\ &\leq [1 + \varepsilon + C (\overline{F}_1(l) + \varepsilon + \varepsilon F_1(l))] (\overline{F}_1(x) + \overline{F}_2(x)). \end{aligned}$$

Relation (2.5) follows because  $\varepsilon$  can be arbitrarily small while  $l$  can be arbitrarily large.  $\square$

**Lemma 2.2.** *Let  $X_1$  and  $X_2$  be two random variables with distributions  $F_1$  and  $F_2$  concentrated on  $[0, \infty)$ , respectively. Under Assumption A1, it holds that*

$$\Pr(X_{(2)} > x) \sim \overline{F}_1(x) + \overline{F}_2(x). \quad (2.7)$$

*Proof.* Clearly,

$$\Pr(X_{(2)} > x) = \overline{F}_1(x) + \overline{F}_2(x) - \Pr(X_1 > x, X_2 > x).$$

Following the steps used to derive (2.3), we obtain that

$$\begin{aligned} \Pr(X_1 > x, X_2 > x) &= \int_x^\infty \frac{\Pr(X_2 > x | X_1 = t)}{\Pr(X_2 > x)} \Pr(X_2 > x) dF_1(t) \\ &\leq \overline{F}_1(x) \overline{F}_2(x) \sup_{x < t < \infty} \frac{\Pr(X_2 > x + t - t | X_1 = t)}{\Pr(X_2 > x + t - t)} \\ &= O(1) \overline{F}_1(x) \overline{F}_2(x). \end{aligned} \quad (2.8)$$

This proves relation (2.7).  $\square$

### 3 The Multivariate Case

We are going to extend Theorem 2.1 to the multivariate case. For recent references on this topic, we refer the reader to Alink *et al.* (2004), Barbe *et al.* (2006), and Kortschak and Albrecher (2008), among others. They all use multivariate copula functions to model the underlying dependence structures.

Let  $X_1, \dots, X_n$  be  $n$  random variables with distributions  $F_1, \dots, F_n$  concentrated on  $[0, \infty)$ , respectively,  $n \geq 2$ . In this case Assumption A1 has to be modified to the following.

*There exists some large  $x_0 > 0$  such that, for every  $j = 2, \dots, n$ , the relation*

$$\frac{\Pr(S_{j-1} > x - t | X_j = t)}{\Pr(S_{j-1} > x - t)} = O(1) \quad \text{Assumption A2}$$

*holds uniformly for all  $t \in [x_0, x]$ .*

Our second main result is given as follows.

**Theorem 3.1.** *Let  $X_1, \dots, X_n$  be  $n$  random variables with distributions  $F_1, \dots, F_n$  concentrated on  $[0, \infty)$ , respectively, such that Assumption A2 holds for all  $j = 2, \dots, n$ . Then the relations*

$$\Pr(S_n > x) \sim \Pr(X_{(n)} > x) \sim \sum_{k=1}^n \overline{F}_k(x) \quad (3.1)$$

hold for each of the following two cases:

1°  $F_k \in \mathcal{S}$  for all  $k = 1, \dots, n$ , and either  $\overline{F}_i(x) = O(\overline{F}_j(x))$  or  $\overline{F}_j(x) = O(\overline{F}_i(x))$  for all  $i, j = 1, \dots, n$ ;

2°  $F_k \in \mathcal{D} \cap \mathcal{L}$  for all  $k = 1, \dots, n$ .

*Proof.* Clearly,

$$\Pr(X_{(n)} > x) \geq \sum_{k=1}^n \overline{F}_k(x) - \sum_{1 \leq i < j \leq n} \Pr(X_i > x, X_j > x).$$

Under Assumption A2, we follow the proof of (2.8) to obtain that, for all  $1 \leq i < j \leq n$ ,

$$\Pr(X_i > x, X_j > x) \leq \Pr(S_{j-1} > x, X_j > x) = o(\overline{F}_j(x)).$$

Therefore,

$$\Pr(X_{(n)} > x) \gtrsim \sum_{k=1}^n \overline{F}_k(x).$$

Since  $S_n \geq X_{(n)}$ , for proving (3.1) it remains to show that

$$\Pr(S_n > x) \lesssim \sum_{k=1}^n \overline{F}_k(x). \quad (3.2)$$

We use mathematical induction on  $n$ . By Lemma 2.1, relation (3.2) holds for  $n = 2$ . Suppose that relation (3.2) holds for  $n-1 \geq 2$ . Hence, relations (3.1) hold for  $n-1$  and the distribution of  $S_{n-1}$  belongs to  $\mathcal{S}$  for case 1° and belongs to  $\mathcal{D} \cap \mathcal{L}$  for case 2°. Furthermore, under the conditions of case 1°, either  $\Pr(S_{n-1} > x) = O(\overline{F}_n(x))$  or  $\overline{F}_n(x) = O(\Pr(S_{n-1} > x))$  holds. Then by Lemma 2.1 again, it immediately follows that

$$\Pr(S_n > x) \lesssim \Pr(S_{n-1} > x) + \overline{F}_n(x) \sim \sum_{k=1}^n \overline{F}_k(x).$$

Thus, relation (3.2) holds for  $n$ . □

We give two special cases of Theorem 3.1 to show that Assumption A2 is verifiable. Similarly as Remarks 2.2 and 2.3, let the random variables  $X_1, \dots, X_n$  be dependent according to a multivariate copula function  $C(u_1, \dots, u_n)$  and let their distributions  $F_1, \dots, F_n$  be absolutely continuous and satisfy the conditions of case 1° or 2° of Theorem 3.1. Assume that the copula density exists:

$$C_{1\dots n}(u_1, \dots, u_n) = \frac{\partial^n}{\partial u_1 \dots \partial u_n} C(u_1, \dots, u_n).$$

**Corollary 3.1.** *Under the above conditions, if, for every nonempty subset  $I$  of  $\{1, \dots, n\}$ , the marginal copula density  $C_I(u_i : i \in I)$  is bounded in a neighborhood of the ultimate vertex (whose coordinates are all 1), then Assumption A2 is fulfilled. Hence, by Theorem 3.1, relations (3.1) hold.*

*Proof.* Let  $c \in (0, 1)$ , and  $M > 0$  be constants such that the inequality

$$C_I(u_i : i \in I) \leq M$$

holds for every nonempty subset  $I$  of  $\{1, \dots, n\}$  and all  $u_i \in [c, 1]$  with  $i \in I$ . Let  $\tilde{x} > 0$  be such that  $F_k(\tilde{x}) \geq c$  for all  $k = 1, \dots, n$  and write  $x_0 = n\tilde{x}$ .

With  $j = 2, \dots, n$  temporarily fixed, let  $I$  be a nonempty subset of  $\{1, \dots, j-1\}$  and write  $I^c = \{1, \dots, j-1\} \setminus I$ . Note that  $C_{I \cup \{j\}}(u_i : i \in I \cup \{j\})$  represents the copula density corresponding to the random variables  $X_i$  for  $i \in I \cup \{j\}$ . We have, for all large  $x$  and all  $x_0 \leq t \leq x - x_0$ ,

$$\begin{aligned} & \Pr(S_{j-1} > x - t, X_i > \tilde{x} \text{ for all } i \in I \text{ and } X_{i^c} \leq \tilde{x} \text{ for all } i^c \in I^c \mid X_j = t) \\ & \leq \Pr\left(\sum_{i \in I} X_i > x - t - x_0, X_i > \tilde{x} \text{ for all } i \in I \mid X_j = t\right). \end{aligned}$$

With  $x_j = t$ , the right-hand side of the above is equal to

$$\int \cdots \int_{(\sum_{i \in I} x_i > x - t - x_0, x_i > \tilde{x} \text{ for all } i \in I)} C_{I \cup \{j\}}(F_i(x_i) : i \in I \cup \{j\}) \prod_{i \in I} dF_i(x_i),$$

which is further bounded by  $M \Pr(\sum_{i \in I} X_i^* > x - t - x_0)$ , where  $(X_1^*, \dots, X_n^*)$  is an independent copy of  $(X_1, \dots, X_n)$ . Therefore, uniformly for all  $t \in [x_0, x - x_0]$ ,

$$\begin{aligned} & \frac{\Pr(S_{j-1} > x - t, X_i > \tilde{x} \text{ for all } i \in I \text{ and } X_{i^c} \leq \tilde{x} \text{ for all } i^c \in I^c \mid X_j = t)}{\Pr(S_{j-1} > x - t)} \\ & \leq \frac{M \Pr(\sum_{i \in I} X_i^* > x - t - x_0)}{\Pr(S_{j-1} > x - t)} \\ & = O(1) \sum_{i \in I} \frac{\Pr(X_i > x - t - x_0)}{\Pr(X_i > x - t)} = O(1). \end{aligned}$$

Summing both sides of the above over all nonempty subsets  $I$  of  $\{1, \dots, j-1\}$  shows that the relation in Assumption A2 holds uniformly for all  $t \in [x_0, x - x_0]$ , or equivalently, uniformly for all  $t \in [x_0, x]$ .  $\square$

**Remark 3.1.** Clearly, a copula whose joint copula density  $C_{1\dots n}(u_1, \dots, u_n)$  is uniformly bounded in the whole domain satisfies the requirements in Corollary 3.1. For example, copulas in the Frank family of the form

$$C(u_1, \dots, u_n; \theta) = -\frac{1}{\theta} \left( 1 + \frac{(e^{-\theta u_1} - 1) \dots (e^{-\theta u_n} - 1)}{(e^{-\theta} - 1)^{n-1}} \right), \quad \theta > 0,$$

as well as copulas in the Clayton family of the form

$$C(u_1, \dots, u_n; \theta) = (u_1^{-\theta} + \dots + u_n^{-\theta} - n + 1)^{-1/\theta}, \quad \theta > 0,$$

belong to this category. □

**Remark 3.2.** It might be tempting to generalize the boundedness assumption in Remark 2.3 to that the joint copula density  $C_{1\dots n}(u_1, \dots, u_n)$  is uniformly bounded on  $[c, 1]^n$  for some  $0 < c < 1$ . In fact, this is not sufficient. As kindly pointed out to us by a referee, the Gaussian copula  $C(u_1, u_2, u_3)$  with the mean vector  $\mu$  and covariance matrix  $\Sigma$  given by

$$\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 1/3 & -2/3 \\ 1/3 & 1 & -2/3 \\ -2/3 & -2/3 & 1 \end{pmatrix},$$

forms such an example. Indeed, one may verify that its joint copula density  $C_{123}(u_1, u_2, u_3)$  is bounded on  $[1/2, 1]^3$  but its marginal copula density  $C_{12}(u_1, u_2)$  is unbounded along the diagonal  $u_1 = u_2$  of  $[1/2, 1]^2$ . □

For case 2° of Theorem 3.1, Assumption A2 can be simplified to the following pairwise version:

*For each pair  $(i, j)$ ,  $1 \leq i < j \leq n$ , and each constant  $c > 0$ , there exists some large  $x_{ij} = x_{ij}(c) > 0$  such that the relation*

$$\frac{\Pr(cX_i > x - t | X_j = t)}{\Pr(cX_i > x - t)} = O(1) \quad \text{Assumption A3}$$

*holds uniformly for all  $t \in [x_{ij}, x]$ .*

**Corollary 3.2.** *For case 2° of Theorem 3.1, Assumption A3 implies Assumption A2. Hence, by Theorem 3.1, relations (3.1) hold under Assumption A3.*

*Proof.* With  $x_0 > 0$  sufficiently large, it holds, for every  $j = 2, \dots, n$ , that

$$\begin{aligned} \sup_{x_0 \leq t \leq x} \frac{\Pr(S_{j-1} > x - t | X_j = t)}{\Pr(S_{j-1} > x - t)} &\leq \sup_{x_0 \leq t \leq x} \sum_{i=1}^{j-1} \frac{\Pr\left(X_i > \frac{x-t}{j-1} \mid X_j = t\right)}{\Pr(X_i > x - t)} \\ &\leq \sum_{i=1}^{j-1} \sup_{x_0 \leq t \leq x} \frac{\Pr\left(X_i > \frac{x-t}{j-1} \mid X_j = t\right) \overline{F}_i\left(\frac{x-t}{j-1}\right)}{\overline{F}_i\left(\frac{x-t}{j-1}\right) \overline{F}_i(x-t)} \\ &= O(1), \end{aligned}$$

where in the last step we used both Assumption A3 and  $F_i \in \mathcal{D}$ . This proves the relation in Assumption A2.  $\square$

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