From Light Tails to Heavy Tails through Multiplier

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Abstract

Let X and Y be two independent nonnegative random variables, of which X has a distribution belonging to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$ and Y is unbounded. We study how their product XY inherits the tail behavior of X. Under some mild technical assumptions we prove that the distribution of XY belongs to the class $\mathcal{L}(0)$ or $\mathcal{S}(0)$ accordingly. Hence, the multiplier Y builds a bridge between light tails and heavy tails.

Keywords: Asymptotics; convolution; discontinuity; multiplier; product; the classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$; upper endpoint.

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1 Introduction and Main Results

Throughout the paper, for a distribution F and for real numbers $x \leq y$, we write $F\{x\} = F(x) - F(x-)$, $F(x, y] = F(y) - F(x)$, $\overline{F}(x) = F(x, \infty) = 1 - F(x)$, and so on. We say that F is supported on $[0, \infty)$ if $F[0, \infty) = 1$ and $\overline{F}(x) > 0$ for all $x \geq 0$.

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{L}(\gamma)$ for some $\gamma \geq 0$ if the relation

$$\lim_{x\to\infty} \frac{\overline{F}\,(x-u)}{\overline{F}(x)} = \mathrm{e}^{\gamma u} \tag{1.1}$$

holds for all u; it is said to belong to the class $\mathcal{S}(\gamma)$ if $F \in \mathcal{L}(\gamma)$ and

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2c \tag{1.2}$$

exists and is finite, where F^{*2} denotes the convolution of F with itself. Since they were introduced by Chistyakov (1964) and Chover et al. (1973a,b) these classes have been extensively investigated and have been applied to many fields of probability theory. It is well known that the constant c in (1.2) is equal to $\int_{0-}^{\infty} \mathrm{e}^{\gamma x} F(\mathrm{d}x)$. Recent studies on these classes are found in

Pakes (2004), Shimura and Watanabe (2005), Tang (2006a), Su and Chen (2006), and Foss and Korshunov (2007), among others. If $\gamma = 0$ then relations (1.1) and (1.2) describe the well-known long-tailed distribution class $\mathcal{L}(0)$ and subexponential distribution class $\mathcal{S}(0)$, respectively.

Recall that a distribution F is said to be rapidly-varying tailed, denoted by $F \in \mathcal{R}_{-\infty}$, if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 0 \qquad \text{for all } y > 1.$$

Recent studies on distributions with rapidly-varying tails can be found in Tang and Tsitsiashvili (2004) and Barbe and McCormick (2008). Clearly, all distributions in the class $\mathcal{L}(\gamma)$ for $\gamma > 0$ belong to the class $\mathcal{R}_{-\infty}$.

Let X and Y be two independent nonnegative random variables with distributions F and G, respectively, and let H be the distribution of their product

$$Z = XY. \tag{1.3}$$

In this paper we study how the product Z inherits the tail behavior of X given that F belongs to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$.

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$, write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, and write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$.

First consider $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$. Definite the (upper) endpoint of Y as

$$\hat{y} = \sup\left\{y : \mathrm{P}(Y \leq y) < 1\right\}.$$

Lemma A.4 of Tang and Tsitsiashvili (2004) shows that if $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$ and $0 < \hat{y} < \infty$ then $H \in \mathcal{L}(\gamma/\hat{y})$. This temps us to conjecture that if $F \in \mathcal{L}(\gamma)$ for $\gamma \geq 0$ and $\hat{y} = \infty$ then $H \in \mathcal{L}(0)$. This is not true, in general. Here is a counterexample:

Example 1.1. Let F be a distribution on $[0, \infty)$ satisfying $F\{1\} > 0$ and $\overline{F}(x) = O(x^{-\alpha})$ for some $\alpha > 0$. Hence for every $\gamma \geq 0$, we can properly construct F so that it belongs to $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$. Let $Y = \mathrm{e}^{\tau}$ with τ being geometric satisfying $\mathrm{P}(\tau = n) = (1-p)p^n$ for some $0 < p < 1$ and all $n = 0, 1, \ldots$. Assume $\alpha + \ln p > 0$. Recall (1.3). Then, for some $c_1, c_2 > 0$,

$$\begin{aligned}\overline{H}(x) &= \left(\sum_{k:1\leq \mathrm{e}^k \leq x} + \sum_{k:\mathrm{e}^k > x}\right) \overline{F}\left(\frac{x}{\mathrm{e}^k}\right)(1-p)p^k \\ &\leq c_1 \sum_{k:1\leq \mathrm{e}^k \leq x} \left(\frac{x}{\mathrm{e}^k}\right)^{-\alpha}(1-p)p^k + \sum_{k:\mathrm{e}^k > x}(1-p)p^k \leq c_2 p^{\ln x}.\end{aligned}$$

For those x such that $x < \mathrm{e}^n \leq x+1$ for some $n = 1, 2, \ldots$, we have

$$\frac{H\left(x, x+1\right]}{\overline{H}(x)} \geq \frac{G\left(x, x+1\right] F\{1\}}{c_2 p^{\ln x}} \geq \frac{(1-p)pF\{1\}}{c_2}.$$

Hence, $H \notin \mathcal{L}(0)$. □

The consequence $H \notin \mathcal{L}(0)$ in Example 1.1 is due to the facts that F is discontinuous at 1 and that $G\,(x, x+1]$ is not negligible in comparison to $\overline{H}(x)$. Denote by $D[F]$ the set of all positive discontinuities of F. Example 1.1 motivates us to assume that the relation

$$G\left(\frac{x}{d}, \frac{x+1}{d}\right] = o(1)\overline{H}(x) \tag{1.4}$$

holds for all $d \in D[F]$ provided that $D[F] \neq \varnothing$. We obtain the following:

Theorem 1.1. *Consider the independent product (1.3) with $F \in \mathcal{L}(\gamma)$ for $\gamma \geq 0$ and $\hat{y} = \infty$. Then, $H \in \mathcal{L}(0)$ if and only if either*

(A). $D[F] = \varnothing$, or

(B). $D[F] \neq \varnothing$ and relation (1.4) holds for all $d \in D[F]$.

In their Theorems 2.1 and 2.2, Su and Chen (2006) have obtained that if $F \in \mathcal{L}(\gamma)$ for $\gamma \geq 0$, $\hat{y} = \infty$, and $D[F] = \varnothing$, then $H \in \mathcal{L}(0)$. We shall give a complete proof of Theorem 1.1 in Section 3. Fortunately, the restriction that relation (1.4) holds for all $d \in D[F]$ is really mild, as illustrated in the result below:

Corollary 1.1. *Consider the independent product (1.3) with $F \in \mathcal{L}(\gamma)$ for $\gamma \geq 0$ and $\hat{y} = \infty$. Then, relation (1.4) holds for all $d > 0$, hence $H \in \mathcal{L}(0)$, in each of the following cases:*

(A). There is some $h > 0$ such that $G(x, x+h]$ is eventually non-increasing in x;

(B). $G \in \mathcal{L}(0)$;

(C). It holds for all $c > 0$ that

$$\lim_{x\to\infty} \frac{\overline{G}(cx)}{\overline{H}(x)} = 0, \tag{1.5}$$

which is further implied by either

(C1). $\overline{G}(vx) = o\left(\overline{G}(x)\right)$ for some $v > 1$, or

(C2). $\overline{G}(vx) = o\left(\overline{F}(x)\right)$ for some $v > 0$.

Proof. We only show the proof for case (A) because the proofs for the other cases are straightforward. For every $d > 0$, there is some positive integer k such that $khd > 1$. We arbitrarily choose an integer n and derive, for all large x,

$$\begin{aligned}
\frac{G\left(\frac{x}{d}, \frac{x+1}{d}\right]}{\overline{H}(x)} &\leq \frac{G\left(\frac{x}{d}, \frac{x}{d} + kh\right]}{\overline{F}\left(\frac{xd}{x-nkhd}\right) G\left(\frac{x}{d} - nkh, \frac{x}{d}\right]} \\
&= \frac{G\left(\frac{x}{d}, \frac{x}{d} + kh\right]}{\overline{F}\left(\frac{xd}{x-nkhd}\right) \sum_{i=1}^{n} G\left(\frac{x}{d} - ikh, \frac{x}{d} - (i-1)kh\right]} \\
&\leq \frac{1}{n\overline{F}\left(\frac{xd}{x-nkhd}\right)} \to \frac{1}{n\overline{F}\left(d\right)},
\end{aligned}$$

Since n can be arbitrarily large, we obtain relation (1.4). □

Theorem 2.2 of Cline and Samorodnitsky (1994) corresponds to the case $\gamma = 0$ of Corollary 1.1(B)(C).

Next consider $F \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$. Theorem 1.1 of Tang (2006a) shows that if $F \in \mathcal{S}(\gamma)$ for $\gamma \geq 0$ and $0 < \hat{y} < \infty$ then $H \in \mathcal{S}(\gamma/\hat{y})$. We aim at an extension of this result to the case $\hat{y} = \infty$. Based on Example 1.1, we see that, in general, the conditions $F \in \mathcal{S}(\gamma)$ for $\gamma \geq 0$ and $\hat{y} = \infty$ can not guarantee $H \in \mathcal{S}(0)$. Theorem 2.1 of Tang (2006b) shows that if $F \in \mathcal{S}(0)$, $\limsup \overline{F}(vx)/\overline{F}(x) < 1$ for some $v > 1$, and relation (1.5) holds for all $c > 0$, then $H \in \mathcal{S}(0)$. Thus, we now only address the case $\gamma > 0$. We obtain the following:

Theorem 1.2. *Consider the independent product (1.3) with $F \in \mathcal{S}(\gamma)$ for $\gamma > 0$ and $\hat{y} = \infty$. If relation (1.5) holds for all $c > 0$ then $H \in \mathcal{S}(0)$.*

Remember that (C) is implied by either (C1) or (C2), as described in Corollary 1.1. We shall prove Theorem 1.2 in Section 4.

2 Two Further Examples

To obtain the subexponentiality of H, a commonly used assumption in the literature is that at least one of F and G is subexponential and the other is relatively light tailed. See Embrechts and Goldie (1980), Cline and Samorodnitsky (1994), Tang (2006b), and Su and Chen (2006). The following example looks somewhat surprising as it does not require the subexponentiality of F or G.

Example 2.1. Consider the independent product (1.3) in which F and G are two exponential distributions with parameters λ_F and λ_G, respectively. We claim that $H \in \mathcal{S}(0)$.

To prove this, we need to derive an asymptotic formula for the tail of H. Since

$$\mathrm{P}\left(Z > x\right) = \mathrm{P}\left(\frac{X}{\lambda_F}\frac{Y}{\lambda_G} > \frac{x}{\lambda_F \lambda_G}\right)$$

in which X/λ_F and Y/λ_G follow the same exponential distribution with parameter 1, without loss of generality we assume that $\lambda_F = \lambda_G = 1$. In this case, we have

$$\overline{H}(x) = \left(\int_0^{\sqrt{x}} + \int_{\sqrt{x}}^{\infty}\right) \mathrm{e}^{-x/y}\mathrm{e}^{-y}\mathrm{d}y.$$

Make change of variables $s = x/y + y - 2\sqrt{x}$ in both integrals above. After some obvious simplification we obtain that

$$\begin{aligned}\overline{H}(x) &= \int_0^{\infty} \mathrm{e}^{-\left(s+2\sqrt{x}\right)}\mathrm{d}\sqrt{s^2 + 4\sqrt{x}s} \\ &= \frac{1}{2}x^{-1/4}\mathrm{e}^{-2x^{1/2}} \int_0^{\infty} s^{1/2}\mathrm{e}^{-s}\frac{\sqrt{4\sqrt{x}s}}{\sqrt{s^2 + 4\sqrt{x}s}}\mathrm{d}s + x^{1/4}\mathrm{e}^{-2x^{1/2}} \int_0^{\infty} s^{-1/2}\mathrm{e}^{-s}\frac{\sqrt{4\sqrt{x}s}}{\sqrt{s^2 + 4\sqrt{x}s}}\mathrm{d}s.\end{aligned}$$

Using the dominated convergence theorem, the two integrals in the last step above converge to $\sqrt{\pi}/2$ and $\sqrt{\pi}$, respectively. It follows that

$$\overline{H}(x) \sim \sqrt{\pi} x^{1/4} \mathrm{e}^{-2x^{1/2}}. \tag{2.1}$$

Based on relation (2.1), applying Theorem 3 of Cline (1986) we see that $H \in \mathcal{S}(0)$. $\square$

Products of random variables are one of basic elements in stochastic modelling in many applied fields. Cline and Samorodnitsky (1994) proposed various potential applications of this study. In addition to these, we point out that products of random variables also naturally appear in multivariate statistical modelling; see, for example, Hashorva (2005). Therefore, it is of fundamental interest to study the tail behavior of the product in (1.3).

To better explain the motivation of the present work, we show another example below, which comes from a rising interdisciplinary area of mathematical finance and actuarial science.

Example 2.2. In recent years, there have been a flurry of papers which focus on the ruin probability of an insurance company making risky investments. The study shows that risky investments may impair the insurer's solvency just as severely as do large claims; see Norberg (1999), Kalashnikov and Norberg (2002), and Tang and Tsitsiashvili (2003).

We use our results to confirm this folklore. Denote by $\{P_t, t \geq 0\}$ the price process of the insurer's investment portfolio, meaning that a unit capital invested at time 0 has an accumulated value P_t at time t, or, equivalently, a unit capital at time t has a present value P_t^{-1} at time 0. Suppose that the first claim of size X comes at time τ. We assume that the random variables X, τ, and the stochastic process $\{P_t, t \geq 0\}$ are mutually independent. With $Y = P_\tau^{-1}$, the present value of this claim is given by

$$Z = P_\tau^{-1} X = XY.$$

We call the random variables X and Y the insurance risk and financial risk, respectively.

For simplicity, we consider a standard Black-Scholes market consisting of a risk-free bond with a constant interest rate $r > 0$ and a risky stock driven by a geometric Brownian motion with instantaneous rate of return $-\infty < b < \infty$ and volatility $\sigma > 0$. Suppose that the insurer continuously invests a constant fraction $\theta \in [0, 1]$ of his wealth in the stock and keeps the remaining wealth in the bond. This constant portfolio is commonly used in mathematical finance; see, for example, Emmer et al. (2001), Klüppelberg and Kostadinova (2008), and references therein. Then, the price process of this investment portfolio is given by

$$P_t = \exp\left\{ \left((1-\theta)\, r + \theta b - \frac{1}{2}\theta^2\sigma^2 \right) t + \theta\sigma W_t \right\}, \qquad t \geq 0, \tag{2.2}$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion.

Let the claim-size distribution F belong to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ for some $\gamma > 0$. If $\theta = 0$, then $Y = P_\tau^{-1} = \exp\{-r\tau\}$, which has an upper endpoint $\hat{y} \in (0, 1]$. Hence by Lemma A.4 of Tang and Tsitsiashvili (2004) and Theorem 1.1 of Tang (2006a), the distribution of

Z belongs to the class $\mathcal{L}(\gamma/\hat{y})$ or $\mathcal{S}(\gamma/\hat{y})$ accordingly. If $0 < \theta \leq 1$, then from (2.2), the financial risk $Y = P_\tau^{-1}$ has an unbounded support. In this case, by Theorems 1.1 and 1.2, the distribution of Z belongs to the class $\mathcal{L}(0)$ or $\mathcal{S}(0)$ accordingly. We conclude that a small fraction of risky investment may significantly enlarge the impact of an insurance claim. This observation matches the common knowledge that risky investments are dangerous, as stated at the beginning of this example. □

3 Proof of Theorem 1.1

We first prepare two elementary analytical results, which play a crucial role in the proof of Theorem 1.1:

Lemma 3.1. *Let f be a monotone finite function on $[a, b]$, possibly with discontinuities of jump size smaller than some $\varepsilon > 0$. Then, there is some $\delta > 0$ such that the inequality*

$$|f(x_1) - f(x_2)| < \varepsilon$$

holds for all $x_1, x_2 \in [a, b]$ satisfying $|x_1 - x_2| < \delta$.

Proof. Using reduction to absurdity, we assume that the claimed result does not hold. Then, we can find a sequence of pairs $\{(x_{1n}, x_{2n}), n = 1, 2, \ldots\}$ satisfying $x_{1n}, x_{2n} \in [a, b]$ and $|x_{1n} - x_{2n}| < 1/n$ such that

$$|f(x_{1n}) - f(x_{2n})| \geq \varepsilon.$$

Hence, there is a subsequence $\{n_k,\ k = 1, 2, \ldots\}$ such that

$$\lim_{k\to\infty} x_{1n_k} = \lim_{k\to\infty} x_{2n_k} = x_0 \in [a, b]$$

but $|f(x_{1n_k}) - f(x_{2n_k})| \geq \varepsilon$ for all k. Define $f(a-) = f(a)$ and $f(b+) = f(b)$. Since f is monotone on $[a, b]$, the sequence $\{f(x_{1n_k}), k = 1, 2, \ldots\}$ has at most two limits, $f(x_0-)$ and $f(x_0+)$, so does the sequence $\{f(x_{2n_k}), k = 1, 2, \ldots\}$. We conclude that

$$|f(x_0+) - f(x_0-)| \geq \varepsilon,$$

contradicting to the assumption. □

Lemma 3.2. *Recall the independent product (1.3). Let relation (1.4) hold for all $d \in D[F]$. Then, for every $M > 0$, every ε satisfying $0 < \varepsilon < \overline{F}(M)$, and all large x,*

$$\int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \geq \left(1 - \frac{\varepsilon}{\overline{F}(M)}\right) \int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \varepsilon \overline{H}(x).$$

Proof. Obviously, F has at most finitely many discontinuities of jump size not less than ε. Only pick such discontinuities falling into the interval $(0, M)$ if any, say x_k, $k = 1, 2, \ldots, n$, with $0 < x_1 < \cdots < x_n < M$. Write $\Delta_x = \bigcup_{k=1}^{n} \left(\frac{x}{x_k}, \frac{x+1}{x_k}\right]$, which is understood as $\varnothing$ if $n = 0$. We have

$$\int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \geq \int_{\left(\frac{x+1}{M},\infty\right)\cap\Delta_x^c} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y). \tag{3.1}$$

In the remaining proof we assume that x is so large that the intervals $\left(\frac{x}{x_k}, \frac{x+1}{x_k}\right]$, $k = 1, 2, \ldots, n$, appearing in the union Δ_x are disjoint and that $\Delta_x \subset \left(\frac{x+1}{M}, \infty\right)$. Note that

$$\left(\frac{x+1}{M}, \infty\right) \cap \Delta_x^c = \left(\frac{x+1}{M}, \frac{x}{x_n}\right] \cup \left(\frac{x+1}{x_n}, \frac{x}{x_{n-1}}\right] \cup \cdots \cup \left(\frac{x+1}{x_2}, \frac{x}{x_1}\right] \cup \left(\frac{x+1}{x_1}, \infty\right).$$

It holds for all large x that

$$\begin{aligned}
&\sup_{y\in\left(\frac{x+1}{M},\infty\right)\cap\Delta_x^c} \left(1 - \frac{\overline{F}\left(\frac{x+1}{y}\right)}{\overline{F}\left(\frac{x}{y}\right)}\right) \\
&\leq \max\left\{\sup_{y\in\left(\frac{x+1}{M},\frac{x}{x_n}\right]}, \sup_{y\in\left(\frac{x+1}{x_n},\frac{x}{x_{n-1}}\right]}, \cdots, \sup_{y\in\left(\frac{x+1}{x_2},\frac{x}{x_1}\right]}, \sup_{y\in\left(\frac{x+1}{x_1},\infty\right)}\right\} \frac{F\left(\frac{x}{y}, \frac{x+1}{y}\right]}{\overline{F}\left(\frac{x}{x+1}M\right)} \\
&\leq \frac{1}{\overline{F}(M)} \max\left\{\sup_{z\in\left[x_n,\frac{x}{x+1}M\right)} F\left(z, z+\frac{M}{x+1}\right], \sup_{z\in\left[x_{n-1},\frac{x}{x+1}x_n\right)} F\left(z, z+\frac{x_n}{x+1}\right],\right. \\
&\qquad\qquad \left.\cdots, \sup_{z\in\left[x_1,\frac{x}{x+1}x_2\right)} F\left(z, z+\frac{x_2}{x+1}\right], \sup_{z\in\left(0,\frac{x}{x+1}x_1\right)} F\left(z, z+\frac{x_1}{x+1}\right]\right\} \\
&\leq \frac{\varepsilon}{\overline{F}(M)},
\end{aligned} \tag{3.2}$$

where in the last step we used Lemma 3.1 and the fact that F has finite left limits at the points M, x_n, $\ldots$, x_1. It follows from (3.1) and (3.2) that

$$\begin{aligned}
\int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) &\geq \left(1 - \frac{\varepsilon}{\overline{F}(M)}\right) \int_{\left(\frac{x+1}{M},\infty\right)\cap\Delta_x^c} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) \\
&\geq \left(1 - \frac{\varepsilon}{\overline{F}(M)}\right) \int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - G(\Delta_x).
\end{aligned}$$

By (1.4), the inequality $G(\Delta_x) \leq \varepsilon\overline{H}(x)$ holds for all large x. This ends the proof. □

PROOF OF THEOREM 1.1:

First we prove the necessity part. Assume $D[F] \neq \varnothing$. For every $d \in D[F]$, we have

$$\begin{aligned}
\overline{H}(x+1) &\leq \left(\int_0^{\infty} - \int_{\frac{x}{d}}^{\frac{x+1}{d}}\right) \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) + \int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \\
&= \overline{H}(x) - \left(\int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y)\right).
\end{aligned} \tag{3.3}$$

Since $\overline{H}(x+1) \sim \overline{H}(x)$, it follows from (3.3) that

$$0 \leq \int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) = o(1)\overline{H}(x). \tag{3.4}$$

Note that the two integrals in (3.4) are asymptotic to $\overline{F}\,(d-)\,G\left(\frac{x}{d}, \frac{x+1}{d}\right]$ and $\overline{F}\,(d)\,G\left(\frac{x}{d}, \frac{x+1}{d}\right]$, respectively. Therefore,

$$\int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \int_{\frac{x}{d}}^{\frac{x+1}{d}} \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \sim F\{d\} G\left(\frac{x}{d}, \frac{x+1}{d}\right]. \tag{3.5}$$

From (3.4) and (3.5) we conclude that relation (1.4) holds.

We then turn to the sufficiency part. The proof below is good for both $D[F] = \varnothing$ and $D[F] \neq \varnothing$. We only need to show that

$$\overline{H}(x+1) \gtrsim \overline{H}(x). \tag{3.6}$$

First assume $\gamma = 0$. Let $0 < \varepsilon_0, \varepsilon_1 < 1$ be arbitrarily fixed. Then, there is some $M > 0$ such that, for all large x and all $\varepsilon_0 < y \leq \frac{x+1}{M}$,

$$\overline{F}\left(\frac{x+1}{y}\right) \geq \overline{F}\left(\frac{x}{y} + \frac{1}{\varepsilon_0}\right) \geq (1-\varepsilon_1)\overline{F}\left(\frac{x}{y}\right). \tag{3.7}$$

Let $0 < \varepsilon_2 < \overline{F}(M)$ also be arbitrarily fixed. By (3.7) and Lemma 3.2 with ε_2 replacing ε, it holds for all large x that

$$\begin{aligned}
\overline{H}(x+1) &\geq \left(\int_{\varepsilon_0}^{\frac{x+1}{M}} + \int_{\frac{x+1}{M}}^{\infty}\right) \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \\
&\geq (1-\varepsilon_1) \int_{\varepsilon_0}^{\frac{x+1}{M}} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) + \left(1 - \frac{\varepsilon_2}{\overline{F}\,(M)}\right) \int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \varepsilon_2 \overline{H}(x) \\
&\geq \min\left\{1-\varepsilon_1, 1 - \frac{\varepsilon_2}{\overline{F}\,(M)}\right\} \int_{\varepsilon_0}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \varepsilon_2 \overline{H}(x).
\end{aligned}$$

Note that

$$\frac{\int_{\varepsilon_0}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y)}{\overline{H}(x)} = \left(1 + \frac{\int_0^{\varepsilon_0} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y)}{\int_{\varepsilon_0}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y)}\right)^{-1} \geq \left(1 + \frac{G(0, \varepsilon_0]}{G(\varepsilon_0, \infty)}\right)^{-1}.$$

It follows that

$$\overline{H}(x+1) \geq \left[\min\left\{1-\varepsilon_1, 1 - \frac{\varepsilon_2}{\overline{F}\,(M)}\right\} \frac{G(\varepsilon_0, \infty)}{G(0, \infty)} - \varepsilon_2\right] \overline{H}(x).$$

Letting $\varepsilon_2 \searrow 0$, $\varepsilon_1 \searrow 0$, and $\varepsilon_0 \searrow 0$, in turn, we obtain relation (3.6).

Next assume $\gamma > 0$. Let $0 < \varepsilon_1 < 1$ and $M_0 > 0$ be arbitrarily fixed. We choose some $M > 0$ such that, for all $y > M_0$ and all $z > M/2$,

$$\frac{\overline{F}\left(z+\frac{1}{y}\right)}{\overline{F}(z)} \geq \frac{\overline{F}\left(z+\frac{1}{M_0}\right)}{\overline{F}(z)} \geq (1-\varepsilon_1)\,\mathrm{e}^{-\frac{\gamma}{M_0}}. \tag{3.8}$$

Similarly to the above, let $0 < \varepsilon_2 < \overline{F}(M)$ also be arbitrarily fixed. By (3.8) and Lemma 3.2 with ε_2 replacing ε, it holds for all large x that

$$\begin{aligned}
\overline{H}(x+1) &\geq \left(\int_{M_0}^{\frac{x+1}{M}} + \int_{\frac{x+1}{M}}^{\infty}\right) \overline{F}\left(\frac{x+1}{y}\right) G(\mathrm{d}y) \\
&\geq (1-\varepsilon_1)\,\mathrm{e}^{-\frac{\gamma}{M_0}} \int_{M_0}^{\frac{x+1}{M}} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) + \left(1-\frac{\varepsilon_2}{\overline{F}(M)}\right) \int_{\frac{x+1}{M}}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \varepsilon_2 \overline{H}(x) \\
&\geq \min\left\{(1-\varepsilon_1)\,\mathrm{e}^{-\frac{\gamma}{M_0}}, 1-\frac{\varepsilon_2}{\overline{F}(M)}\right\} \int_{M_0}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) - \varepsilon_2 \overline{H}(x). \qquad (3.9)
\end{aligned}$$

Since $F \in \mathcal{R}_{-\infty}$, by Lemma A.3 of Tang and Tsitsiashvili (2004) we have

$$\int_{M_0}^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) \sim \overline{H}(x).$$

Substituting this into (3.9), then letting $\varepsilon_2 \searrow 0$, $\varepsilon_1 \searrow 0$, and $M_0 \nearrow \infty$, in turn, we again obtain relation (3.6). □

4 Proof of Theorem 1.2

The following is a variant form of Lemma 3.2 of Tang (2006a):

Lemma 4.1. *Let X_1 and X_2 be two i.i.d. nonnegative random variables with common distribution $F \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$. Then for arbitrarily fixed $0 < \delta < 1$ and $M > 0$, the relation*

$$\mathrm{P}\left(X_1 + sX_2 > x\right) \lesssim \mathrm{e}^{\gamma M/\delta}\overline{F}\left(\frac{x}{s \vee \delta}\right) + \left(2\int_M^{\infty} \mathrm{e}^{\gamma u} F(\mathrm{d}u) + \mathrm{E}\left[\exp\left\{(s\vee\delta)\,\gamma X_1\right\}\right]\right)\overline{F}(x)$$

holds uniformly over all $s \in [0,1]$.

PROOF OF THEOREM 1.2:

It suffices to prove that

$$\overline{H^{*2}}(x) \lesssim 2\overline{H}(x) \tag{4.1}$$

since $\overline{H^{*2}}(x) \gtrsim 2\overline{H}(x)$ is automatic for all distributions H on $[0,\infty)$. Let (X_i, Y_i), $i = 1, 2$, be i.i.d. copies of (X, Y). Under the condition that relation (1.5) holds for all $c > 0$, Lemma 3.2

of Tang (2006b) shows that there is a function $a(\cdot):[0,\infty)\to[0,\infty)$ satisfying $a(x)\nearrow\infty$, $a(x)/x\searrow 0$, and $\overline{G}(a(x))=o(1)\overline{H}(x)$. In terms of this function we derive

$$\begin{aligned}
&\overline{H^{*2}}(x)\\
&\le \mathrm{P}(X_1Y_1+X_2Y_2>x,Y_2\le Y_1\le a(x))+\mathrm{P}(X_1Y_1+X_2Y_2>x,Y_1<Y_2\le a(x))+2\overline{G}(a(x))\\
&= I_1(x)+I_2(x)+o(1)\overline{H}(x).
\end{aligned}$$

Conditioning on (Y_1,Y_2) and applying Lemma 4.1 with arbitrarily fixed $0<\delta<1$ and $M>0$,

$$\begin{aligned}
&I_1(x)\\
&= \iint\limits_{0\le y_2\le y_1\le a(x)} \mathrm{P}\left(X_1+\frac{y_2}{y_1}X_2>\frac{x}{y_1}\right)G(\mathrm{d}y_1)G(\mathrm{d}y_2)\\
&\lesssim \mathrm{e}^{\gamma M/\delta}\iint\limits_{0\le y_2\le y_1\le a(x)}\overline{F}\left(\frac{x/y_1}{(y_2/y_1)\vee\delta}\right)G(\mathrm{d}y_1)G(\mathrm{d}y_2)\\
&\quad+\iint\limits_{0\le y_2\le y_1\le a(x)}\left(2\int_M^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u)+\mathrm{E}\left[\exp\left\{\left(\frac{y_2}{y_1}\vee\delta\right)\gamma X\right\}\right]\right)\overline{F}\left(\frac{x}{y_1}\right)G(\mathrm{d}y_1)G(\mathrm{d}y_2).
\end{aligned}$$

A similar relation holds for $I_2(x)$. Therefore,

$$\begin{aligned}
\overline{H^{*2}}(x)&\lesssim \mathrm{e}^{\gamma M/\delta}\mathrm{P}\left(X\left((Y_1\wedge Y_2)\vee\delta Y_1\vee\delta Y_2\right)>x\right)\\
&\quad+\iint\limits_{0\le y_2\le y_1}\left(2\int_M^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u)+\mathrm{E}\left[\exp\left\{\left(\frac{y_2}{y_1}\vee\delta\right)\gamma X\right\}\right]\right)\overline{F}\left(\frac{x}{y_1}\right)G(\mathrm{d}y_1)G(\mathrm{d}y_2)\\
&\quad+\iint\limits_{0\le y_1< y_2}\left(2\int_M^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u)+\mathrm{E}\left[\exp\left\{\left(\frac{y_1}{y_2}\vee\delta\right)\gamma X\right\}\right]\right)\overline{F}\left(\frac{x}{y_2}\right)G(\mathrm{d}y_1)G(\mathrm{d}y_2)\\
&\quad+o(1)\overline{H}(x)\\
&= J_1(x)+J_2(x)+J_3(x)+o(1)\overline{H}(x).
\end{aligned}\tag{4.2}$$

Clearly,

$$J_2(x)+J_3(x)\ge \mathrm{E}\left[\exp\{\delta\gamma X\}\right]\mathrm{P}\left(X\left(Y_1\vee Y_2\right)>x\right),\tag{4.3}$$

whereas

$$J_1(x)\le \mathrm{e}^{\gamma M/\delta}\left(\mathrm{P}\left(X\left(Y_1\wedge Y_2\right)>x\right)+2\mathrm{P}\left(\delta XY>x\right)\right).\tag{4.4}$$

Since $\hat{y}=\infty$, by Lemma 3.3 of Tang (2006a) we have

$$\mathrm{P}\left(X\left(Y_1\wedge Y_2\right)>x\right)=o(1)\mathrm{P}\left(X\left(Y_1\vee Y_2\right)>x\right).\tag{4.5}$$

Furthermore, since $F\in\mathcal{R}_{-\infty}$ and relation (1.5) holds for all $c>0$, it is not difficult to see that $H\in\mathcal{R}_{-\infty}$ too. Hence,

$$\mathrm{P}\left(\delta XY>x\right)=o(1)\mathrm{P}\left(XY>x\right)=o(1)\mathrm{P}\left(X\left(Y_1\vee Y_2\right)>x\right).\tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) and using (4.3), we have

$$J_1(x) = o(1)\mathrm{P}\left(X\left(Y_1 \vee Y_2\right) > x\right) = o(1)\left(J_2(x) + J_3(x)\right). \tag{4.7}$$

Therefore, it follows from (4.2), (4.3), and (4.7) that

$$\overline{H^{*2}}(x) \lesssim J_2(x) + J_3(x). \tag{4.8}$$

For arbitrarily fixed $\varepsilon > 0$, choose some M in (4.8) such that $2\int_M^\infty \mathrm{e}^{\gamma u} F(\mathrm{d}u) \le \varepsilon \mathrm{E} \exp\{\delta\gamma X\}$. Then, for this $\varepsilon > 0$ and all $y_0 > 0$,

$$\begin{aligned}\overline{H^{*2}}(x) &\lesssim (1+\varepsilon)\sum_{i=1}^{2}\int_0^\infty \mathrm{E}\left[\exp\left\{\left(\frac{Y}{y_i} \vee \delta\right)\gamma X\right\} 1_{(Y/y_i \le 1)}\right] \overline{F}\left(\frac{x}{y_i}\right) G(\mathrm{d}y_i)\\ &\sim (1+\varepsilon)\sum_{i=1}^{2}\int_{y_0}^\infty \mathrm{E}\left[\exp\left\{\left(\frac{Y}{y_i} \vee \delta\right)\gamma X\right\} 1_{(Y/y_i \le 1)}\right] \overline{F}\left(\frac{x}{y_i}\right) G(\mathrm{d}y_i),\end{aligned}$$

where the last step can be verified by the fact that $F \in \mathcal{R}_{-\infty}$. Hence,

$$\begin{aligned}\overline{H^{*2}}(x) &\lesssim (1+\varepsilon)\sup_{y_0<y<\infty} \mathrm{E}\left[\exp\left\{\left(Y/y \vee \delta\right)\gamma X\right\} 1_{(Y/y \le 1)}\right] \sum_{i=1}^{2}\int_{y_0}^\infty \overline{F}\left(\frac{x}{y_i}\right) G(\mathrm{d}y_i)\\ &\le 2(1+\varepsilon)\sup_{y_0<y<\infty} \mathrm{E}\left[\exp\left\{\left(Y/y \vee \delta\right)\gamma X\right\} 1_{(Y/y \le 1)}\right] \overline{H}(x).\end{aligned}$$

By the dominated convergence theorem,

$$\lim_{\delta\searrow 0}\lim_{y_0\nearrow\infty}\sup_{y_0<y<\infty} \mathrm{E}\left[\exp\left\{\left(Y/y \vee \delta\right)\gamma X\right\} 1_{(Y/y \le 1)}\right] = \lim_{\delta\searrow 0} \mathrm{E}\left[\exp\left\{\delta\gamma X\right\}\right] = 1.$$

We obtain that $\overline{H^{*2}}(x) \lesssim 2(1+\varepsilon)\overline{H}(x)$. This proves relation (4.1). □

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