Combining Non-Stably Infinite Theories

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Abstract

The Nelson-Oppen combination method combines decision procedures for first-order theories over disjoint signatures into a single decision procedure for the union theory. To be correct, the method requires that the component theories be stably infinite. This restriction makes the method inapplicable to many interesting theories such as, for instance, theories having only finite models.

In this paper we provide a generalization of the Nelson-Oppen combination method that can combine any theory that is not stably infinite with another theory, provided that the latter is what we call a *shiny* theory. Examples of shiny theories include the theory of equality, the theory of partial orders, and the theory of total orders.

An interesting consequence of our results is that any decision procedure for the satisfiability of quantifier-free Σ-formulae in a Σ-theory T can always be extended to accept inputs over an arbitrary signature $\Omega \supseteq \Sigma$.

Keywords: Theory reasoning, multiple background reasoners.

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1 Introduction

An important research problem in automated reasoning asks how we can modularly combine decision procedures for theories T_1 and T_2 into a decision procedure for a combination of T_1 and T_2.

The most successful and well known method for combining decision procedures was invented in 1979 by Nelson and Oppen [NO79]. This method is at the heart of the verification systems CVC [SBD02], ESC [DLNS98], EVES [CKM$^+$91], and SDVS [LFMM92], among others.

The Nelson-Oppen method allows us to decide the satisfiability of quantifier-free formulae in a combination T of a theory T_1 and a theory T_2, using as black boxes a decision procedure for the satisfiability of quantifier-free formulae in T_1 and a decision procedure for the satisfiability of quantifier-free formulae in T_2.

The method is correct whenever the theories T, T_1, and T_2 satisfy the following restrictions:

- T is logically equivalent to $T_1 \cup T_2$;
- the signatures of T_1 and T_2 are disjoint;
- T_1 and T_2 are both stably infinite.[1]

There are several interesting combination problems that do not satisfy all these restrictions.

In this paper we concentrate on the issue of relaxing the stable infiniteness requirement. This is an important research problem at the theoretical level because it allows us to better understand the foundations of combination problems, and to prove more decidability results by combination techniques. But it is also interesting at a practical level because (i) proving that a given theory is stably infinite is not always easy, and (ii) many interesting theories, such as those admitting only finite models, are not stably infinite.

We show that when one component theory satisfies a stronger property than stable infiniteness, which we call *shininess*,[2] then the other component theory does not need to be stably infinite for their decision procedures to be combinable. We do that by providing and proving correct a variant of the Nelson-Oppen method that, in addition to propagating equality constraints between the component decision procedures as in the original method, also propagates certain cardinality constraints.

In providing examples of shiny theories, we show that theory of equality, the theory of partial orders, and the theory of total orders are shiny. In particular, the fact that the theory of equality is shiny leads to a notable side result:

> **Result 1.** *If the satisfiability in a Σ-theory T of quantifier-free Σ-formulae is decidable, then the satisfiability in T of quantifier-free formulae over any arbitrary signature $\Omega \supseteq \Sigma$ is also decidable.*

[1] See Definition 4.
[2] See Definition 8.

In particular, the fact that the theory of equality is shiny leads to a notable side result which, to our knowledge, has never been proven before. The side result is that if the satisfiability in a Σ-theory T of quantifier-free Σ-formulae is decidable, then the satisfiability in T of quantifier-free formulae over any arbitrary signature $\Omega \supseteq \Sigma$ is also decidable, regardless of whether T is stably infinite or not. proven by Policriti and Schwartz [PS95] for theories T that are universal. It was also known for theories T that are stably infinite, since in this case one can use the Nelson-Oppen method to combine the decision procedure for T with one for the theory of equality over the symbols in $\Omega \setminus \Sigma$. In this paper we prove that Result 1 holds regardless of whether T is universal/stably infinite or not.

1.1 Related work

Several researchers have worked on relaxing the requirements of the Nelson-Oppen combination method. The disjointness problem was addressed by Ghilardi [Ghi03], Tinelli [Tin03], Tinelli and Ringeissen [TR03] and Zarba [Zar02c]. The stably infiniteness requirement was addressed by Baader and Tinelli [BT97] for combinations problems concerning the word problem, and by Zarba [Zar01, Zar02a, Zar02b] for combinations of integers with lists, sets, and multisets. (The latter works by Zarba consider combination problems other than simple set-theoretic union.)

1.2 Organization of the paper

The paper is organized as follows. In Section 2 we introduce some preliminary notions that we will use in the paper, including the notion of a shiny theory. In Section 3 we describe a modification of the Nelson-Oppen combination method for combining the decision procedure of a shiny theory with that of any other arbitrary theory. In Section 4 we provide some examples that illustrate the behavior of our extension, and contrast it with the Nelson-Oppen method. In Section 5 we prove that our method is correct. In Section 6 we present some examples of shiny theories We conclude in Section 7 with directions for further research.

2 Preliminaries

2.1 Syntax

A *signature* Σ is composed by a set Σ^{C} of constants, a set Σ^{F} of function symbols, and a set Σ^{P} of predicate symbols.

A Σ-*atom* is either an expression of the form $P(t_1, \ldots, t_n)$, where $P \in \Sigma^{\mathrm{P}}$ and $t_1, \ldots, t_n$ are Σ-terms, or an expression of the form $s \approx t$, where $\approx$ is the equality logical symbol and s, t are Σ-terms. Σ-*formulae* are constructed by applying in the standard way the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and the quantifiers

$\forall, \exists$ to Σ-atoms. Σ-*literals* are Σ-atoms or their negations. Σ-*sentences* are Σ-formulae with no free variables.

If φ is a term or a formula, $\mathit{vars}(\varphi)$ denotes the set of variables occurring in φ. Similarly, if Φ is a set of terms or a set of formulae, $\mathit{vars}(\Phi)$ denotes the set of variables occurring in Φ.

In the rest of this paper, we identify a conjunction of formulae $\varphi_1 \wedge \cdots \wedge \varphi_n$ with the set $\{\varphi_1, \ldots, \varphi_n\}$. In addition, we abbreviate literals of the form $\neg(s \approx t)$ with $s \not\approx t$.

2.2 Semantics

Definition 1. Let Σ be a signature. A Σ-INTERPRETATION $\mathcal{A}$ with domain A over a set of variables V is a map which interprets:

- each variable x as an element $x^{\mathcal{A}} \in A$;
- each constant $c \in \Sigma^{\mathrm{C}}$ as an element $c^{\mathcal{A}} \in A$;
- each function symbol $f \in \Sigma^{\mathrm{F}}$ of arity n as a function $f^{\mathcal{A}} : A^n \to A$;
- each predicate symbol $P \in \Sigma^{\mathrm{P}}$ of arity n as a subset $P^{\mathcal{A}}$ of A^n. □

Unless otherwise specified, we use the convention that calligraphic letters $\mathcal{A}$, $\mathcal{B}$, ... denote interpretations, and that the corresponding Roman letters A, B, ... denote the domains of the interpretations.

Let $\mathcal{A}$ be a Σ-interpretation over a set of variables V. For a Σ-term t over V, we denote with $t^{\mathcal{A}}$ the evaluation of t under the interpretation $\mathcal{A}$. Likewise, for a Σ-formula φ over V, we denote with $\varphi^{\mathcal{A}}$ the truth-value of φ under the interpretation $\mathcal{A}$. If T is a set of Σ-terms over V, we denote with $T^{\mathcal{A}}$ the set $\{t^{\mathcal{A}} \mid t \in T\}$.

A formula φ is *satisfied* by an interpretation $\mathcal{A}$ if it evaluates to true under $\mathcal{A}$. If φ is satisfied by $\mathcal{A}$, we say that $\mathcal{A}$ is a *model* of φ. A formula φ over a set V of variables is:

- *valid*, if it is satisfied by all interpretations over V;
- *satisfiable*, if it is satisfied by some interpretation over V;
- *unsatisfiable*, if it is not satisfiable.

The notion of validity, satisfiability, and unsatisfiability naturally extend to sets of formulae.

Definition 2. Let Σ be a signature, and let $\mathcal{A}$ and $\mathcal{B}$ be Σ-interpretations over some set V of variables. A map $h : A \to B$ is an EMBEDDING of $\mathcal{A}$ into $\mathcal{B}$ if the following conditions hold:

- h is injective;
- $h(u^{\mathcal{A}}) = u^{\mathcal{B}}$ for each variable or constant $u \in V \cup \Sigma^{\mathrm{C}}$;

- $h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n))$, for each n-ary function symbol $f \in \Sigma^{\mathrm{F}}$ and $a_1, \ldots, a_n \in A$;
- $(a_1, \ldots, a_n) \in P^{\mathcal{A}}$ if and only if $(h(a_1), \ldots h(a_n)) \in P^{\mathcal{B}}$, for each n-ary predicate symbol $P \in \Sigma^{\mathrm{P}}$ and $a_1, \ldots, a_n \in A$. □

2.3 Theories

Definition 3. Let Σ be a signature. A Σ-THEORY is any set of Σ-sentences. □

A theory T is *axiomatized* by a set S of sentences if S and T are logically equivalent.

Given a Σ-theory T, a *T-model* is a Σ-interpretation that satisfies all sentences in T. A formula φ over a set V of variables is:

- *T-valid*, if it is satisfied by all T-models over V;
- *T-satisfiable*, if it is satisfied by some T-model over V;
- *T-unsatisfiable*, if it is not T-satisfiable.

The notion of T-validity, T-satisfiability, and T-unsatisfiability naturally extend to sets of formulae.

Given a Σ-theory T and a set L of formulae, the *satisfiability problem* of T with respect to L is the problem of deciding, for each formula φ in L, whether or not φ is T-satisfiable. When we do not specify L, it is implicitly assumed that L is the set of all Σ-formulae. However, when we say "quantifier-free satisfiability problem", without specifying L, then we implicitly assume that L is the set of all quantifier-free Σ-formulae.

In this paper, we will use the usual notion of stable infiniteness for a theory, together with its "dual" one, which we call stable finiteness.

Definition 4. A Σ-theory T is STABLY INFINITE (respectively, STABLY FINITE) if every quantifier-free Σ-formula φ is T-satisfiable if and only if it is satisfied by a T-interpretation $\mathcal{A}$ whose domain A is infinite (respectively, finite). □

Examples of stably infinite theories include the theory of equality,[3] the theory of integer arithmetic, the theory of rational arithmetic, the theory of acyclic lists, and the theory of arrays.

Examples of stably finite theories include the theory of equality, all theories satisfied only by finite interpretations, all theories axiomatized by formulas in the Bernays-Schönfinkel-Ramsey class, and all theories whose axioms do not contain $\approx$.

Note that a theory can be both stably infinite and stably infinite. We will show that in Section 6.1 for the theory of equality.

[3] Since we regard $\approx$ as a logical symbol, for us the theory of equality and the empty theory are the same theory.

Definition 5. A Σ-theory T is SMOOTH if for every quantifier-free Σ-formula φ, for every T-model $\mathcal{A}$ satisfying φ, and for every cardinal number $\kappa > |A|$ there exists a T-model $\mathcal{B}$ satisfying φ such that $|B| = \kappa$. □

The following proposition is a direct consequence of Definition 5.

Proposition 6. *Every smooth theory is stably infinite.* □

The following proposition is useful when proving that a theory is smooth.

Proposition 7. *Let T be a Σ-theory. Then the following are equivalent:*

1. *T is smooth;*
2. *for every quantifier-free Σ-formula φ and for every finite T-model $\mathcal{A}$ of φ there exists a T-model $\mathcal{B}$ of φ such that $|B| = |A| + 1$.* □

PROOF. $(1 \Rightarrow 2)$. Trivial.

$(2 \Rightarrow 1)$. Let φ be a quantifier-free formula, and let $\mathcal{A}$ be a model of φ.

By induction on $|A|$, one can see that if A is finite then φ has a model of any finite cardinality $\kappa > |A|$. By compactness, φ has a countably infinite model, and by the Upward Löwenheim-Skolem Theorem, φ has also a model of any infinite cardinality κ.

If instead A is infinite then, by the upward Löwenheim-Skolem Theorem, it follows that φ has a model of any (infinite) cardinality $\kappa > |A|$. ■

Given a theory T and a T-satisfiable quantifier-free formula φ, we denote with $mincard_T(\varphi)$ the smallest cardinality of a T-model satisfying φ. When T is the theory of equality, we abbreviate $mincard_T$ with $mincard$ (without any subscript).

Note that if T is a stably finite theory then, for every T-satisfiable formula φ, $mincard_T(\varphi)$ is a natural number.

Definition 8. A Σ-theory T is SHINY if:

- T is smooth;
- T is stably finite;
- $mincard_T$ is computable. □

3 The combination method

Let S be a shiny Σ-theory and let T be an Ω-theory such that $\Sigma \cap \Omega = \emptyset$ and the quantifier-free satisfiability problem of S and of T is decidable. We now describe a method for combining decision procedures for the quantifier-free

satisfiability problems of S and T into a single decision procedure for quantifier-free satisfiability problem of $S \cup T$.

Without loss of generality, we restrict ourselves to conjunctions of literals. Note that this can always be done because every formula φ can be effectively converted into an equisatisfiable formula in disjunctive normal form $\psi_1 \vee \cdots \vee \psi_n$, where each ψ_i is a conjunction of literals. Then φ is satisfiable if and only if at least one of the disjuncts ψ_i is satisfiable.

The combination method consists of four phases, described below.

First phase: variable abstraction

Let Γ be a conjunction of $(\Sigma \cup \Omega)$-literals. In this phase we convert Γ into a conjunction Γ' satisfying the following properties:

(a) each literal in Γ' is either a Σ-literal or an Ω-literal;

(b) Γ' is $(S \cup T)$-satisfiable if and only if so is Γ.

Note that all properties can be effectively enforced with the help of new auxiliary variables.

Second phase: partition

Let Γ' be a conjunction of literals obtained in the variable abstraction phase. In the second phase we rewrite Γ' as $\Gamma_1 \cup \Gamma_2$ where:

- Γ_1 contains all Σ-literals in Γ';
- Γ_2 contains all Ω-literals in Γ'.

We call $\Gamma_1 \cup \Gamma_2$ a conjunction of literals in *separate* form.

Third phase: decomposition

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a conjunction of literals obtained in the partition phase. Let V be the set of variables shared by Γ_1 and Γ_2, that is $V = \mathit{vars}(\Gamma_1) \cap \mathit{vars}(\Gamma_2)$.

In this phase we nondeterministically guess an equivalence relation E over V. Intuitively, what we are guessing is, for each variable $x, y \in V$, whether or not we have $x = y$.

Fourth phase: check

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a conjunction of literals in separate form, let $V = \mathit{vars}(\Gamma_1) \cap \mathit{vars}(\Gamma_2)$, and let E be the equivalence relation over V guessed in the decomposition phase. The fourth phase consists in performing the following steps:

Step 1. Construct the *arrangement* of V induced by E, defined by

$$\begin{aligned} \mathit{arr}(V, E) = \{x \approx y \mid x, y \in V, x, y \text{ are distinct and } (x, y) \in E\} \cup \\ \{x \not\approx y \mid x, y \in V \text{ and } (x, y) \notin E\}. \end{aligned}$$

Step 2. If $\Gamma_1 \cup arr(V, E)$ is S-satisfiable go to the next step; otherwise output `fail`.

Step 3. Compute $n = mincard_S(\Gamma_1 \cup arr(V, E))$.

Step 4. Construct a set δ_n of literals whose purpose is to force models with cardinality at least n. More precisely, let δ_n be the set of literals constructed with the following process:

- generate n new variables $w_1, \ldots, w_n$ not occurring in $\Gamma_1 \cup \Gamma_2$;
- let $\delta_n = \{w_i \not\approx w_j \mid 1 \leq i < j \leq n\}$.

Step 5. If $\Gamma_2 \cup arr(V, E) \cup \delta_n$ is T-satisfiable output `succeed`; otherwise output `fail`.

In Section 5 we will prove that:

- if there exists an equivalence relation E over V for which the check phase outputs `succeed` then Γ is $(S \cup T)$-satisfiable;
- if instead the check phase outputs `fails` for each equivalence relation E over V, then Γ is $(S \cup T)$-unsatisfiable.

The combination method above is a variant of the non-deterministic version of the Nelson-Oppen combination method [Rin96, TH96]. The only substantial differences are in the fourth phase above: In the Nelson-Oppen method, Step 3 and 4 are absent, and Step 5 checks the T satisfiability of $\Gamma_2 \cup arr(V, E)$ only. Note that this is enough in the Nelson-Oppen method because there we assume that T is stably infinite, and therefore the constraint δ_n is guaranteed to hold.

Note that our method applies just as well in case T is stably-infinite.[4] However, if one knows that T is stably infinite, resorting to the original Nelson-Oppen method is more appropriate, as it lets one avoid the cost of computing $mincard_S$.

4 Examples

In this section we discuss two examples of theories that are not combinable with the Nelson-Oppen method but are combinable with our method. In both examples we combine the theory S of equality over a signature Σ with a non-stably infinite theory T over a signature Ω disjoint from Σ. In the first case, T is not stably infinite because it only admits finite models. In the second case, T is not stably infinite even if it has infinite models. The examples are adapted from [TH96] and [BT97], respectively, where they are used to show that the Nelson-Oppen method is in fact incorrect on non-stably infinite theories.

[4] Recall that S is already stably infinite, since it is shiny.

Example 9. Let $\Sigma = \{f\}$ and $\Omega = \{g\}$ be signatures, where f and g are distinct unary function symbols. Also, let S be the theory of equality over the signature Σ, and let T be an Ω-theory such that all T-interpretations have cardinality at most two.

Since T is not stably infinite, we cannot use the Nelson-Oppen combination method in order to combine S with T. However, in Section 6.1 we will show that the theory of equality is shiny, regardless of the associated signature. Thus, we can apply the method described in the previous section to S and T.

As an example, let Γ be the following conjunction of literals:

$$\Gamma = \left\{ \begin{array}{l} f(x) \not\approx f(y)\,, \\ f(x) \not\approx f(z)\,, \\ g(y) \not\approx g(z) \end{array} \right\}.$$

Note that Γ is $(S \cup T)$-unsatisfiable. In fact, Γ implies $x \not\approx y \wedge x \not\approx z \wedge y \not\approx z$, and therefore every interpretation satisfying Γ must have cardinality at least three. Since every $(S \cup T)$-interpretation has at most two elements, it follows that Γ is $(S \cup T)$-unsatisfiable.

Let us apply our combination method to Γ. In the variable abstraction phase we do not need to generate any new variables. In the partition phase we simply return the conjunctions

$$\Gamma_1 = \left\{ \begin{array}{l} f(x) \not\approx f(y)\,, \\ f(x) \not\approx f(z) \end{array} \right\}, \qquad \Gamma_2 = \left\{ \; g(y) \not\approx g(z) \; \right\}.$$

Since $vars(\Gamma_1) \cap vars(\Gamma_2) = \{y, z\}$, in the check phase there are only two equivalence relations to examine: either $(y, z) \in E$ or not $(y, z) \notin E$.

If $(y, z) \in E$ we have that $\Gamma_1 \cup \{y \approx z\}$ is S-satisfiable and that $\Gamma_2 \cup \{y \approx z\}$ is T-unsatisfiable. Thus, we will output `fail` when reaching step 4 of the check phase.

If instead $(y, z) \notin E$ then $\Gamma_1 \cup \{y \not\approx z\}$ is S-satisfiable. In addition, we have $mincard_S(\Gamma_1 \cup \{y \not\approx z\}) = 3$. To see this, first observe that $\Gamma_1 \cup \{y \not\approx z\}$ implies $x \not\approx y \wedge x \not\approx z \wedge y \not\approx z$, and therefore $mincard_S(\Gamma_1 \cup \{y \not\approx z\}) \geq 3$. In addition, we can construct an interpretation $\mathcal{A}$ of cardinality 3 satisfying $\Gamma_1 \cup \{y \not\approx z\}$ by letting $A = \{a_1, a_2, a_3\}$, $x^{\mathcal{A}} = a_1$, $y^{\mathcal{A}} = a_2$, $z^{\mathcal{A}} = a_3$, and $f^{\mathcal{A}}(a) = a$, for each $a \in A$.[5]

In the third step of the check phase we introduce three new variables w_1, w_2, w_3, and construct δ_3 as the set $\{w_1 \not\approx w_2, w_1 \not\approx w_3, w_2 \not\approx w_3\}$. Since $\Gamma_2 \cup \{y \not\approx z\} \cup \delta_3$ is T-unsatisfiable, in the fourth step we output `fail`. We can therefore declare that Γ is $(S \cup T)$-unsatisfiable.

Note that since the Nelson-Oppen method checks the T-satisfiability of just $\Gamma_2 \cup \{y \not\approx z\}$ (and not of $\Gamma_2 \cup \{y \not\approx z\} \cup \delta_3$), it may *incorrectly* output `succeed` on input Γ, because $\Gamma_2 \cup \{y \not\approx z\}$ is satisfiable in a model of cardinality 2. □

Example 10. Let $\Sigma = \{k\}$ and $\Omega = \{f, g, h\}$ be signatures, where k, f and g are distinct unary function symbols. Let S be again the theory of equality over

[5] We will see how to effectively compute $mincard_S$ in Section 6.1.

the signature Σ, and let T be the equational theory

$$T = \left\{ \begin{array}{l} (\forall x)(\forall y)(x \approx f(g(x), g(y))), \\ (\forall x)(\forall y)(f(g(x), h(y)) \approx y) \end{array} \right\}.$$

Using simple term rewriting arguments, it is possible to show that T admits models of cardinality greater than one, and so admits models of infinite cardinality.[6] However, T is not stably infinite.

In fact, consider the set quantifier-free formula $g(z) \approx h(z)$. This formula is T-satisfiable because both the formula and T admit a trivial model, i.e. a model with just one element. Now let $\mathcal{A}$ be any T-model of $g(z) \approx h(z)$, let $a_0 = z^{\mathcal{A}}$, and let $a \in A$. Because of T's axioms we have that

$$a = f^{\mathcal{A}}(g^{\mathcal{A}}(a), g^{\mathcal{A}}(a_0)) = f^{\mathcal{A}}(g^{\mathcal{A}}(a), h^{\mathcal{A}}(a_0)) = a_0$$

Given that a is arbitrary, this entails that $|A| = 1$. Thus, $g(z) \approx h(z)$ is only satisfiable in trivial models of T, which immediately entails that the theory T is not stably infinite.

For an application of our combination method to S and T, let Γ be the following conjunction of literals:

$$\Gamma = \left\{ \begin{array}{l} g(z) \approx h(z)\,, \\ k(z) \not\approx z \end{array} \right\}.$$

This conjunction is $(S \cup T)$-unsatisfiable, because $g(z) \approx h(z)$ is satisfiable only in trivial models of $S \cup T$ (for being satisfiable only in trivial models of T, as seen above), while $k(z) \not\approx z$ is clearly satisfiable only in non-trivial models of $S \cup T$.

Let us apply our combination method to Γ. In the partition phase we simply return the conjunctions

$$\Gamma_1 = \left\{ \; k(z) \not\approx z \; \right\}, \qquad \Gamma_2 = \left\{ \; g(z) \approx h(z) \; \right\}.$$

Since $\mathit{vars}(\Gamma_1) \cap \mathit{vars}(\Gamma_2) = \{z\}$, in the check phase there are no equivalence relations to examine, therefore we generate the empty arrangement. Clearly, Γ_1 is S-satisfiable, and in models of cardinality at least 2. Therefore, we have that $\mathit{mincard}_S(\Gamma_1) = 2$.

In the third step of the check phase, we then compute δ_2 as the set $\{w_1 \not\approx w_2\}$ for some fresh variables w_1, w_2. For what we argued above, $\Gamma_2 \cup \delta_2$ is T-unsatisfiable, so in the fourth step we output `fail`, as needed. □

5 Correctness

In this section we prove that our combination method is correct.

Clearly, our combination method is terminating. This follows from the fact that, since there is only a finite number of equivalence relations over a finite set

[6] This is because the set of models of an equational theory is closed under direct products.

V of variables, the nondeterministic decomposition phase is finitary. Thus, we only need to prove that our method is partially correct.

We will use the following theorem which is a special case of a more general combination result given in [TR03] for theories with possibly non-disjoint signatures. A direct proof of this theorem can be found in [MZ03].

Theorem 11 (Combination Theorem for Disjoint Signatures). *Let Φ_i be a set of Σ_i-formulae, for $i = 1, 2$, and let $\Sigma_1 \cap \Sigma_2 = \emptyset$.*

Then $\Phi_1 \cup \Phi_2$ is satisfiable if and only if there exists an interpretation $\mathcal{A}$ satisfying Φ_1 and an interpretation $\mathcal{B}$ satisfying Φ_2 such that:

(i) $|A| = |B|$,

(ii) $x^{\mathcal{A}} = y^{\mathcal{A}}$ *if and only if* $x^{\mathcal{B}} = y^{\mathcal{B}}$, *for every* $x, y \in vars(\Phi_1) \cap vars(\Phi_2)$. □

The following proposition proves that our combination method is also partially correct.

Proposition 12. *Let S be a shiny Σ-theory and let T be an Ω-theory such that $\Sigma \cap \Omega = \emptyset$. Let Γ_1 be a conjunction of Σ-literals and let Γ_2 be a conjunction of Ω-literals. Finally, let $V = vars(\Gamma_1) \cap vars(\Gamma_2)$. Then, the following are equivalent:*

1. *$\Gamma_1 \cup \Gamma_2$ is $(S \cup T)$-satisfiable*
2. *there exists an equivalence relation E of V such that:*
   - *(i) $\Gamma_1 \cup arr(V, E)$ is S-satisfiable;*
   - *(ii) $\Gamma_2 \cup arr(V, E) \cup \delta_n$ is T-satisfiable where $n = mincard_S(\Gamma_1 \cup arr(V, E))$.* □

Proof. We will assume, as in the combination method, that the variables of the formula δ_n are fresh.

$(1 \Rightarrow 2)$. Assume that $\Gamma_1 \cup \Gamma_2$ is $(S \cup T)$-satisfiable, and let $\mathcal{F}$ be one of its $(S \cup T)$-models. Let

$$E = \{(x, y) \mid x, y \in V \text{ and } x^{\mathcal{F}} = y^{\mathcal{F}}\}.$$

Clearly, $\mathcal{F}$ is an $(S \cup T)$-model of $\Gamma_1 \cup \Gamma_2 \cup arr(E, V)$. It follows that $\mathcal{F}$ is also an S-model of $\Gamma_1 \cup arr(E, V)$, thus proving (i). In addition, $\mathcal{F}$ is a T-model of $\Gamma_2 \cup arr(E, V)$.

Let $\kappa = |F|$, and let $n = mincard_S(\Gamma_1 \cup arr(V, E))$. By definition of $mincard_S$, we have $n \leq \kappa$, which implies that $\mathcal{F}$ is also a T-model of $\Gamma_2 \cup arr(E, V) \cup \delta_n$, proving (ii).

$(2 \Rightarrow 1)$. Let $V_1 = vars(\Gamma_1)$ and $V_2 = vars(\Gamma_2 \cup \delta_n)$, and observe that $V_1 \cap V_2 = V$. Assume there is an equivalence relation E of V such that $\Gamma_1 \cup arr(V, E)$ is

S-satisfiable and $\Gamma_2 \cup arr(V,E) \cup \delta_n$ is T-satisfiable, where $n = mincard_S(\Gamma_1 \cup arr(V,E))$.

Then there exist an S-model $\mathcal{A}$ of $\Gamma_1 \cup arr(V,E)$ and a T-model $\mathcal{B}$ of $\Gamma_2 \cup arr(V,E) \cup \delta_n$.

Since $\mathcal{B}$ satisfies δ_n, we have $|B| \geq n$. Thus, by the smoothness of S, we can assume without loss of generality that $|A| = |B|$. In addition, because both $\mathcal{A}$ and $\mathcal{B}$ satisfy $arr(V,E)$, we have that $x^{\mathcal{A}} = y^{\mathcal{A}}$ if and only if $x^{\mathcal{B}} = y^{\mathcal{B}}$, for all $x, y \in V$.

By the Combination Theorem for Disjoint Signatures 11, $S \cup T \cup \Gamma_1 \cup \Gamma_2 \cup arr(V,E) \cup \delta_n$ is satisfiable. Thus, $\Gamma_1 \cup \Gamma_2$ is $(S \cup T)$-satisfiable. ■

Combining Proposition 12 with the fact that our combination method is terminating, we obtain the following decidability result.

Theorem 13 (Decidability). *Let S be a shiny Σ-theory and let be T an Ω-theory such that $\Sigma \cap \Omega = \emptyset$. If the quantifier-free satisfiability problems of S and of T are decidable, then the quantifier-free satisfiability problem of $S \cup T$ is also decidable.* □

6 Applications

In this section, we present some examples of shiny theories to which our combination results apply: the theory of equality, the theory of partial orders, and the theory of total orders.

6.1 The theory of equality

It is well known that the theory of equality (over an arbitrary signature) is stably infinite and has a decidable quantifier-free satisfiability problem [Opp80].

We show here that the theory of equality is also shiny. To do that we will use the following basic lemma of model theory, adapted from page 44 of Hodges's book [Hod97] .

Lemma 14. *Let $\mathcal{A}, \mathcal{B}$ be two interpretations such that there is an embedding of $\mathcal{A}$ into $\mathcal{B}$, and let φ be a quantifier-free formula. Then φ is satisfied by $\mathcal{A}$ if and only if it is satisfied by $\mathcal{B}$.* □

Proposition 15. *Let φ be a quantifier-free formula, and let $\mathcal{A}$ be a finite model of φ. Then there exists a model $\mathcal{B}$ of φ such that $|B| = |A| + 1$.* □

Proof. Let $k = |A|$. We construct a Σ-model $\mathcal{B}$ of φ such that $|B| = k + 1$ as follows. Let

$$B = A \cup \{b\},$$

where $b \notin A$. Then, fix an arbitrary element $a_0 \in B$, and let

Input: An S-satisfiable conjunction Γ of Σ-literals
Output: *mincard*(Γ)

```
 1: if Γ is empty then
 2:    return 1
 3: else
 4:    U ← TERMS(Γ)
 5:    Γ′ ← Γ
 6:    for s, t ∈ U do
 7:       if Γ′ ∪ {s ≈ t} is S-satisfiable then
 8:          Γ′ ← Γ′ ∪ {s ≈ t}
 9:       end if
10:    end for
11:    E ← {(s, t) | s ≈ t ∈ Γ′}
12:    C ← CONG-CLOSURE(E)
13:    return CARD(U/C)
14: end if
```

Figure 1: A procedure for *mincard*.

- for variables and constants:

$$u^{\mathcal{B}} = u^{\mathcal{A}}\,,$$

- for function symbols of arity n:

$$f^{\mathcal{B}}(a_1,\ldots,a_n) = \begin{cases} f^{\mathcal{A}}(a_1,\ldots,a_n)\,, & \text{if } a_1,\ldots,a_n \in A\,,\\ a_0\,, & \text{otherwise,}\end{cases}$$

- for predicate symbols of arity n:

$$(a_1,\ldots,a_n) \in P^{\mathcal{B}} \iff a_1,\ldots,a_n \in A \text{ and } (a_1,\ldots,a_n) \in P^{\mathcal{A}}\,.$$

We have $|B| = k+1$. In addition, the map $h : A \to B$ defined by $h(a) = a$, for each $a \in A$, is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Since $\mathcal{A}$ satisfies φ, by Lemma 14 it follows that $\mathcal{B}$ also satisfies φ. ∎

Combining Proposition 7 and 15, we obtain the smoothness of the theory of equality.

Proposition 16. *For every signature Σ, the Σ-theory of equality is smooth.* □

Next, we show that *mincard*(φ) is computable for any satisfiable quantifier-free formula φ. A procedure that computes *mincard* is given in Figure 1. For simplicity, and without of generality, the procedure takes as input only satisfiable conjunctions of literals, returning a positive integer.

In the procedure, the function TERMS returns the set of all terms and subterms occurring in its input Γ. For instance, if $\Gamma = \{f(g(x)) \approx g(f(y))\}$

then TERMS(Γ) returns the set $\{x, g(x), f(g(x)), y, f(y), g(f(y))\}$. The function CONG-CLOSURE computes the congruence closure of the binary relation E over the signature of Γ.[7] $U/_C$ denotes the quotient of U with respect to the congruence relation C.

Both C and $U/_C$ can be computed using any standard congruence closure algorithm [DST80, Koz77, NO80, Sho78]. The complexity of such algorithms is $\mathcal{O}(n^2)$, where n is the cardinality of U. The test in line 7 can be performed by the same congruence closure algorithm used for computing C. Since the procedure is clearly terminating, it then follows that its complexity is $\mathcal{O}(n^4)$.

We show below that the procedure is also partially correct.

Proposition 17. *For every input Γ, the procedure shown in Figure 1 returns $mincard(\Gamma)$.* □

PROOF. If Γ is empty then Γ is satisfied by every interpretation. Thus, in this case the procedure returns the correct value $mincard(\Gamma) = 1$.

Let us consider the case in which Γ is not empty. Let U, Γ', E and C be as computed by the procedure. Moreover, let k be the value returned in line 13. Note that Γ' is S-satisfiable, and that $\Gamma \subseteq \Gamma'$. Thus, every S-model of Γ' is also an S-model of Γ. Finally, since Γ is not empty, then U is not empty either. It follows that the quotient $U/_C$ is also not empty, hence $k \geq 1$.

Let $\mathcal{A}$ be any S-model of Γ', and consider the set

$$B = \{t^{\mathcal{A}} \mid t \in U\}\,.$$

We claim that $|B| = k$. To see this, suppose, for a contradiction, that $|B| \neq k$. Then either $|B| < k$ or $|B| > k$.

Assume first that $|B| < k$. Since k is equal to the number of equivalence classes of C, there exist two terms $s, t \in U$ such that $(s,t) \notin C$ and $s^{\mathcal{A}} = t^{\mathcal{A}}$. But then $\Gamma' \cup \{s \approx t\}$ is satisfied by $\mathcal{A}$, which implies that $s \approx t \in \Gamma'$. It follows that $(s,t) \in E$, and therefore $(s,t) \in C$, a contradiction.

Next, suppose that $|B| > k$. Then there exist distinct terms $t_1, \ldots, t_n$, with $n > k$, such that $t_i^{\mathcal{A}} \neq t_j^{\mathcal{A}}$, for $i < j$. Since C is the congruence closure of E, it follows that, for every term s, t, if $(s,t) \in C$ then $s^{\mathcal{A}} = t^{\mathcal{A}}$. But then, for every term s, t, if $s^{\mathcal{A}} \neq t^{\mathcal{A}}$ then $(s,t) \notin C$. Thus, $(t_i, t_j) \notin C$, for $i < j$. It follows that C has more than k equivalence classes, a contradiction.

Since $|B| = k$, by the generality of $\mathcal{A}$, we can conclude that every S-model of Γ has at least k elements.

We now construct an S-model $\mathcal{B}$ of Γ with domain B. The proposition's claim will then follow from the fact that $|B| = k$.

Let b be some element of B. We define

- for variables and constants:

$$u^{\mathcal{B}} = \begin{cases} u^{\mathcal{A}}\,, & \text{if } u^{\mathcal{A}} \in B\,, \\ b\,, & \text{otherwise}\,, \end{cases}$$

[7] Given a binary relation E, the congruence closure of E is the smallest congruence C containing E.

- for function symbols of arity n:

$$f^{\mathcal{B}}(b_1,\ldots,b_n) = \begin{cases} f^{\mathcal{A}}(b_1,\ldots,b_n)\,, & \text{if } f^{\mathcal{A}}(b_1,\ldots,b_n) \in B\,, \\ b\,, & \text{otherwise,} \end{cases}$$

- for predicate symbols of arity n:

$$(b_1,\ldots,b_n) \in P^{\mathcal{B}} \quad \Longleftrightarrow \quad (b_1,\ldots,b_n) \in P^{\mathcal{A}}\,.$$

We have that $\mathcal{B}$ is an S-interpretation. Moreover, by structural induction, one can show that $t^{\mathcal{B}} = t^{\mathcal{A}}$ for all terms $t \in U$, and that $\ell^{\mathcal{B}} = \ell^{\mathcal{A}}$ for all literals $\ell \in \Gamma'$. It follows that $\mathcal{B}$ satisfies Γ'. Since $\Gamma \subseteq \Gamma'$, $\mathcal{B}$ also satisfies Γ. ■

As an immediate corollary of Proposition 17, we obtain the following result.

Proposition 18. *For every signature Σ, the Σ-theory of equality is stably finite.* □

Putting together Propositions 7, 17, and 18, we obtain the shininess of the theory of equality.

Proposition 19. *For every signature Σ, the Σ-theory of equality is shiny.* □

Proposition 19 is relevant because, together with our combination method in Section 3, it tells us that any procedure that decides the quantifier-free satisfiability problem for a Σ-theory T can be extended to accept inputs Γ containing arbitrary free symbols[8] in addition to the symbols in Σ. This fact was already known for theories T that are universal [PS95]. It was also known for theories T that are stably-infinite, since in this case one can use the Nelson-Oppen method to combine the decision procedure for T with one for the theory of equality over the symbols of Γ that are not in Σ. Thanks to Proposition 19 and our combination method, we are able to lift the universal and/or stable-infiniteness requirement for T altogether.

More formally, we have the following theorem.

Theorem 20. *Let T be a Σ-theory such that the quantifier-free satisfiability problem of T is decidable. Then, for every signature $\Omega \supseteq \Sigma$, the quantifier-free satisfiability problem of T with respect to Ω-formulae is decidable.* □

6.2 BSR-theories

In this subsection we show that a large class of theories immediately satisfy all the requirements for being combinable with our method except smoothness. Among these theories we single out a couple, as an example, that are in fact also smooth.

We call these theories *BSR-theories* after Bernays, Schönfinkel, and Ramsey, who studied some of their properties.

[8] Also referred to as "uninterpreted" symbols by some authors.

Definition 21 (BSR-theories). A sentence φ is a BSR-SENTENCE if it is of the form $(\exists x_1)\cdots(\exists x_m)(\forall y_1)\cdots(\forall y_n)\psi$, where $m, n \geq 0$ and ψ is a quantifier-free formula that does not contain function symbols.

A BSR-THEORY is a finite set of BSR-sentences. □

The following proposition was proved by Bernays and Schönfinkel [BS28] for the case of first-order logic without equality, and by Ramsey [Ram30] for the case of first-order logic with equality.

Proposition 22. *Let Φ a conjunction of BSR-sentences. Then there exists an integer k, bounded above by the size of Φ, such that Φ is satisfiable if and only if it has a model of cardinality at most k.* □

An immediate consequence of Proposition 22 is that the satisfiability of finite sets Φ of BSR-sentences is decidable: one simply Skolemizes Φ into a set Φ' and checks the satisfiability of Φ' in Herbrand interpretations. Now, it is easy to see that Φ' will contain no function symbols, therefore all Herbrand interpretations over the signature of Φ' are finite. It is enough then to construct all such interpretations up to cardinality k until one is found that satisfies Φ'.

Proposition 22 is interesting to us because it entails that the quantifier-free satisfiability problem of any BSR-theory T is decidable. The reason is simply that a quantifier-free formula φ is T-satisfiable exactly when the finite set $T \cup \varphi'$ of BSR-sentences is satisfiable, where φ' is the existential closure of φ. From this observation it is also immediate that the following proposition holds.

Proposition 23. *Every BSR-theory T is stably finite. Moreover, the quantifier-free satisfiability problem of T is decidable and $mincard_T$ is computable.* □

The discussion above suggests a general algorithm for computing $mincard_T(\varphi)$ when T is a BSR-theory. The complexity of this algorithm is bounded by the number of Σ-interpretations over $vars(\varphi)$ of cardinality at most n, where n is the size of the conjunction $T \cup \{\varphi\}$. Since there is an exponential number of such interpretations, the algorithm belongs to the class of complexity EXPTIME. Particular theories of course may admit a much faster algorithm, specific to that theory.

Note that, in general, BSR-theories are not smooth, and so not shiny either. For instance, the theory

$$T = \{\ (\forall x)(\forall y)(x \approx y)\ \}$$

is a BSR-theory, but it is obviously not smooth because it only admits models of cardinality 1. We provide some examples of smooth BSR-theories in the next two subsections.

Partial and total orders

We now provide two examples of shiny BSR-theories, the theories of partial and of total orders. The theory PO of partial orders is defined by the following

axioms:

$$(\forall x)\neg(x < x) \qquad \text{(irreflexivity)}$$
$$(\forall x)(\forall y)(\forall z)(x < y \ \wedge \ y < z \ \rightarrow \ x < z) \qquad \text{(transitivity)}\,.$$

The theory TO of total orders extends the theory of partial orders, with the following axiom

$$(\forall x)(\forall y)(x < y \ \vee \ x = y \ \vee \ y < x) \qquad \text{(trichotomy)}\,.$$

Since both PO and TO are BSR-theories, they are both stably finite. Moreover, both $mincard_{PO}$ and $mincard_{TO}$ are computable. We prove that both PO and TO are also smooth.

Proposition 24. *Let $\Sigma = \{<\}$, let φ be a quantifier-free Σ-formula, let $\mathcal{A}$ be a finite PO-model of φ, and let $k = |A|$ be a natural number. Then there exists a PO-model $\mathcal{B}$ of φ of cardinality $k+1$.* □

Proof. We construct a Σ-model $\mathcal{B}$ of φ such that $|B| = k+1$ as follows. Let

$$B = A \cup \{b\}\,,$$

where $b \notin A$. Then let

$$u^{\mathcal{B}} = u^{\mathcal{A}}\,, \qquad \text{for variables and constants}\,,$$

and

$$a_1 <^{\mathcal{B}} a_2 \quad \Longleftrightarrow \quad a_1 <^{\mathcal{A}} a_2 \text{ and } a_1, a_2 \in A\,.$$

We have $|B| = k+1$. In addition, the map $h : A \rightarrow B$ defined by $h(a) = a$, for each $a \in A$, is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Since $\mathcal{A}$ satisfies φ, by Lemma 14 it follows that $\mathcal{B}$ also satisfies φ. ■

Combining Proposition 7 and 24 we obtain the smoothness of the theory of partial orders.

Proposition 25. *The theory PO of partial orders is smooth.* □

Proposition 26. *Let $\Sigma = \{<\}$, let φ be a quantifier-free Σ-formula, let $\mathcal{A}$ be a finite TO-model of φ, and let $k = |A|$ be a natural number. Then there exists a TO-model $\mathcal{B}$ of φ of cardinality $k+1$.* □

Proof. We construct a Σ-model $\mathcal{B}$ of φ such that $|B| = k+1$ as follows. Let

$$B = A \cup \{b\}\,,$$

where $b \notin A$. Then let

$$u^{\mathcal{B}} = u^{\mathcal{A}}\,, \qquad \text{for variables and constants}\,,$$

and

$$a_1 <^{\mathcal{B}} a_2 \quad \Longleftrightarrow \quad \left[\begin{array}{c} a_1 <^{\mathcal{A}} a_2 \text{ and } a_1, a_2 \in A \\ \text{or} \\ a_1 \neq b \text{ and } a_2 = b \end{array} \right]$$

Intuitively, we defined $<^{\mathcal{B}}$ exactly as $<^{\mathcal{A}}$, with the difference that the new element b becomes the maximum element in the total order $<^{\mathcal{B}}$.

We have $|B| = k + 1$. In addition, the map $h : A \to B$ defined by $h(a) = a$, for each $a \in A$, is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Since $\mathcal{A}$ satisfies φ, by Lemma 14 it follows that $\mathcal{B}$ also satisfies φ. ■

Combining Proposition 7 and 26 we obtain the smoothness of the theory of total orders.

Proposition 27. *The theory TO of total orders is smooth.* □

In conclusion, we have the following results.

Proposition 28. *The theory TO of total orders and the theory PO partial orders are shiny.* □

Theorem 29. *Where O is either TO or PO, let T be any theory signature-disjoint with O. If the quantifier-free satisfiability problem of T is decidable, then the quantifier-free satisfiability problem of $O \cup T$ is also decidable.* □

7 Conclusion

We have addressed the problem of extending the Nelson-Oppen combination method to theories that are not stably infinite. We provided a modification of the Nelson-Oppen method for combining two theories, in which it is possible to lift the stable infiniteness requirement from one theory, provided that the other one satisfies a stronger condition, which we called shininess.

We gave some examples of shiny theories, namely the theory of equality, the theory of partial orders, and the theory of total orders.

In particular, the shininess of the theory of equality yields an interesting useful result: Any decision procedure for the quantifier-free satisfiability problem of a theory T can always be extended to accept input formulae over an arbitrary signature. The usefulness of this result stems from the fact that, in practice, satisfiability problems in a theory T often contain free function symbols in addition to the original symbols of T.[9] Our result says that these symbols can be always dealt with properly, no matter what T is.

The Nelson-Oppen method is applicable to an arbitrary number of stably infinite and pairwise signature-disjoint theories. Similarly, our method can be extended to the combination of one arbitrary theory and $n > 1$ shiny theories, all pairwise signature-disjoint. It is unlikely that our method be extended to

[9] These function symbols are typically introduced by skolemization or abstraction processes.

allow more than one arbitrary theory. In fact, if this were the case, we would be able to combine two arbitrary theories.

The correctness proof of both the Nelson-Oppen method and our method relies on the Combination Theorem for Disjoint Theories (Theorem 11). That theorem requires that the two parts of a separate form of an input formula be satisfied in models of the respective theories having the same cardinality. As pointed out in [TR03], this requirement is impossible to check in general. Considering only stably infinite theories, as done in the original method, allows one to completely forgo the check, because stably infinite theories always satisfy it. Our method deals with the cardinality requirement by assuming enough on one theory, the shiny one, so that a simpler cardinality check, the one represented by δ_n, can be performed on the other.

We plan to continue our reserch on relaxing the stable infiniteness requirement by aiming at finding general sufficient conditions for shininess, and at identifying additional specific examples of shiny theories.

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