Outline for Lecture 1: First properties of affine varieties

Denote the base field by k.

I. First definition of a variety: An affine algebraic variety is the zero locus of a collection of polynomials.

CAREFUL! Many people (including Hartshorne) call this an algebraic set.

II. FACT: (One version of Hilbert's Nullstellensatz) If k is algebraically closed, there is a bijective correspondance

$$\text{varieties} \leftrightarrow \textit{radical} \text{ ideals}$$
$$\text{zero locus of } F_1, \ldots, F_m \leftrightarrow \text{ideal } \sqrt{(F_1, \ldots, F_m)}$$

Definition 0.1. *The radical* $\sqrt{I}$ *of the ideal* I *is the collection of roots of elements of* I*:*

$$\sqrt{I} = \{f : f^m \in I \text{ for some } m\}.$$

Notation: The ideal corresponding to a variety Y is written $I(Y)$.
Notation: The variety of points vanishing on an ideal I is written $V(I)$.

This correspondance reverses order:

$$I(Y_1) \subseteq I(Y_2) \Leftrightarrow Y_1 \supseteq Y_2.$$

III. Geometric consequences of algebraic correspondance:

$$Y \text{ is a point} \leftrightarrow Y \text{ is maximal}$$
$$Y \text{ is irreducible} \leftrightarrow Y \text{ is prime}$$

A variety Y is irreducible if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two nonempty subvarieties (neither of which equal Y). Intuitively, "irreducible" means "has only one piece."

1. First properties of affine varieties

Over the course of these lectures, we will give an overview of algebraic geometry, discussing the main objects in the field as well as some of the big questions driving research today. Our intended audience is the interested outsider, or the tourist who needs a quick primer on the essential ideas. Other texts, off which we have liberally ripped, treat the subject more in-depth and rigorously. Those readers who leave these notes wanting more are directed, in no particular order, to the books of Hartshorne, Shafarevich, Griffiths and Harris, Eisenbud and Harris, Harris all alone, or Reid.

What is algebraic geometry? It's the field that studies geometry and algebra, exploiting the interplay between the two. Let's outline both of these topics separately.

Geometry describes geometric objects. We already have an intuitive understanding of some of the ways we can describe a geometric object. For instance, we could ask for:

(1) the dimension of the object,
(2) whether the object is singular or smooth,
(3) the number of connected components, or
(4) the number of irreducible components (number of essentially different pieces).

We'll defer any deeper discussion of what each of these questions might mean. In general, while learning about algebraic geometry, we rely on intuition in ways that can be uncomfortable for those who think rigorously; simultaneously, we rely on abstract definitions in ways that can be uncomfortable for those who think intuitively. This makes the initial stages of learning algebraic geometry a bit uncomfortable for just about everybody.

What puts the **algebra** in algebraic geometry is that the objects we study are defined by polynomials. For instance, when thinking geometrically, we'll consider a circle in the plane; when thinking algebraically, we'll consider the solutions to the polynomial $x^2 + y^2 - 1 = 0$. The key point is that we will not use $\mathcal{C}^1$ functions or $\mathcal{C}^\infty$ functions or analytic functions.

First definition of a variety: An affine algebraic variety is the zero locus of a collection of polynomials.

CAREFUL! Many people (including Hartshorne) call this an algebraic set.

The first **question** this definition raises is: zero locus where? For example, consider the equation $x^2 + y^2 = 0$.

(1) If you look at its solutions in $\mathbb{R}^2$, you get the single point $\{(0,0)\}$.
(2) If you look at its solutions in $\mathbb{C}^2$, you get the pair of lines $x = \pm iy$.

We won't answer this question once and for all. Instead, we'll work over the field k. One of the recurring motifs of the field is the difference that k makes.

Note: Often when graphing I'll draw $\mathbb{R}$ for $\mathbb{C}$.

We are now ready to define algebraic varieties more rigorously.

Definition 1.1. Algebraic Variety: *Let k be a field. Let $\mathcal{F}$ be a collection of polynomials in $k[x_1, ..., x_n]$. Then the affine algebraic variety associated to $\mathcal{F}$ is*

$$V(\mathcal{F}) = \{p \in \mathbb{A}^n \;:\; f(p) = 0 \text{ for all } f \in \mathcal{F}\}.$$

Note: $\mathbb{A}^n$ refers to affine n-dimensional k-space. (When we really want to stress the dependence on k, we'll write $\mathbb{A}^n_k$.) For most practical purposes, you could think of this as k^n. The difference is that we don't assume any vector space structure. In particular, the origin is just the same as any other point in $\mathbb{A}^n$. Indeed, a basic trick in algebraic geometry—so common it barely warrants the name 'trick'—is to pick a random point p and then assume p is the point $(0, 0, \dots, 0)$. The notational convenience is clear, and since it just required translation in affine space (or an affine change of coordinates $x \mapsto x + p$), the trick didn't alter any fundamental algebro-geometric properties of the underlying variety.

Other possible fields are $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$, or even $\mathbb{Z}$, which isn't a field at all. These examples bring us to the intersection of algebraic geometry and number theory, where one of the most celebrated recent results appeared.

Fermat's Last Theorem: Does the variety defined by $x^n + y^n = z^n$ have any integer-valued points?

Our convention will usually be that k is $\mathbb{C}$ or at least algebraically closed. The algebraic geometer you meet on the street is probably a complex algebraic geometer. Number theorists are more interested in other fields, so we will continue to specify our assumptions on k. Algebraic closure is useful because it ensures good intersection—or factorization!—properties. For instance, take the famous

Theorem 1.2. Fundamental Theorem of Algebra: *If $f(x) \in \mathbb{C}[x]$ has degree n then it has n roots (with multiplicity).*

What it says about algebraic varieties is

Theorem 1.3. Geometric Interpretation of the Fundamental Theorem of Algebra: *The variety associated to $f = 0$ has n points over $\mathbb{C}$ (with multiplicity).*

We now give several curves in $\mathbb{R}[x, y]$. These will be some of the examples we use again and again, partly because they are easily drawn.

Example 1.4. Varieties:

(1) $y^2 = x^2(x + 1)$ *is known as the nodal cubic.*
(2) $y^2 = x^3$ *is known as the cuspidal cubic.*
(3) $xy = 0$
(4) $x = 0$

Up to this point, we have been considering the question: What geometric object is defined by a collection of polynomials? We now turn out attention to the converse.

Question: Which polynomials define a given geometric object?

There are two main ways to get redundancy in the defining set of polynomials.

(1) The polynomial $x^2y^2 = 0$ defines the same variety as $xy = 0$.
(2) We can also add polynomials that do not "remove" any elements from the variety. For instance, $\{nxy \ : \ n \in \mathbb{N}\}$ defines the same variety as xy.

We'll address the second kind of redundancy first. In general, if g is generated by $f_1, ..., f_r$ over $k[x_1, ..., x_n]$ then $\{f_1, ...f_r, g\}$ and $\{f_1, ..., f_r\}$ define the same variety. If we want to find some canonical set of polynomials associated to a particular variety, our first thought might be to look for a minimal generating set. Unfortunately, this works poorly: each of the following pairs of polynomials

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x + y = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x - y = 0 \\ x + y = 0 \end{cases}$$

define the same variety. The truly persistent may prefer one among these sets, but you can see that choosing a minimal generating set in general will require a lot of arbitrary and, what's worse, inelegant choices. (Those interested in algorithms to choose small generating sets might wish to look into *Gröbner bases.*)

We've established that there's no easy way to pick a minimal set of polynomials that defines a given variety. So instead, we'll pick a maximal generating set!

Definition 1.5. $I(X)$**:** *Suppose that X is a variety. Then the ideal associated to X is*

$$I(X) = \{f \in k[x_1, ..., x_n] \;:\; f(p) = 0 \text{ for all } p \in X\}$$

considered as an ideal in the polynomial ring.

The following are all basic examples.

Example 1.6. Ideals of a given variety.

(1) *Ideal of k^2 in $k[x, y]$?* (0)
(2) *Ideal of $\emptyset$ in $k[x, y]$?* $(1) = k[x, y]$
(3) *Ideal of the origin $(0, 0)$?* (x, y)
(4) *Ideal of x-axis in $k[x, y]$?* (y)
(5) *Ideal of coordinate axes in $k[x, y]$?* (xy)

The observant reader may observe a general principal through these examples: as the variety gets bigger, the ideal gets smaller, and vice versa. The next proposition states this more formally. Its proof is a good exercise, and is basically a problem in set theory.

Proposition 1.7. *If Y_1 and Y_2 are two varieties then $I(Y_1) \supseteq I(Y_2) \Leftrightarrow Y_1 \subseteq Y_2$.*

To recap, we can associate ideals to varieties and varieties to ideals.

$$\begin{array}{ll} \text{Ideal to Variety by} & X \longmapsto I(X) \\ \text{Variety to Ideal by} & I \longmapsto V(I) = \{p \in \mathbb{A}^n \;:\; f(p) = 0 \;\; \forall f \in I\}. \end{array}$$

Is this correspondence a bijection? The answer is no, but in two very different ways.

(1) If k is not algebraically closed there are problems. We saw this phenomenon earlier: the variety associated to $(x^2 + y^2)$ in $\mathbb{R}^2$ is just the origin, which is also the variety associated to the ideal (x, y). This can easily be remedied by assuming k is algebraically closed.

(2) More problematically, we have $V((x^2)) = V((x))$, even if k is algebraically closed. We can fix this problem, too, but it takes a new definition.

Definition 1.8. Radical of an Ideal: *The radical of an ideal I is*

$$\sqrt{I} = \{\text{polynomials } f : f^m \in I \text{ for some } m\}.$$

An ideal I is called radical *if $I = \sqrt{I}$.*

This is exactly the tool we need to have a bijective correspondence between ideals and varieties.

Theorem 1.9. Hilbert Nullstellensatz (one version): *If k is algebraically closed, then the correspondence $V \longleftrightarrow I(V)$ is a bijection between algebraic varieties and radical ideals.*

The Nullstellensatz lets us immediately identify the geometric objects corresponding to some of our favorite ideals: prime and maximal ideals.

Corollary 1.10. *Under the above-mentioned bijective correspondence, points correspond to maximal ideals.*

The proof is immediate from what we've said already (though make sure you believe me). Identifying prime ideals again requires a new definition, one that gets at the notion of "substantively different pieces" of a connected variety.

Definition 1.11. Irreducible Variety: *A variety Y is irreducible if it cannot be written as $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are two non-empty, proper subvarieties.*

Theorem 1.12. *The variety Y is irreducible if and only if $I(Y)$ is prime.*

Proof. We will show $I(Y)$ is not prime iff Y is reducible.

($\Longrightarrow$) If $I(Y)$ is not prime then there exists $f, g \notin I(Y)$ such that $fg \in I(Y)$. Each of $V(f, I(Y))$ and $V(g, I(Y))$ are proper subvarieties of Y since by hypothesis neither f nor g vanishes on all of Y. Moreover $Y = V(f, I(Y)) \cup V(g, I(Y))$ since fg vanishes for every point in Y. Together, these imply that neither $V(f, I(Y))$ nor $V(g, I(Y))$ are empty, so Y is reducible.

($\Longleftarrow$) Assume that $Y = Y_1 \cup Y_2$ is a decomposition demonstrating that Y is reducible. In particular, we know neither $Y_1 \subseteq Y_2$ nor $Y_2 \subseteq Y_1$. From the relation $I(Y_1) \supseteq I(Y_2) \Leftrightarrow Y_1 \subseteq Y_2$, we conclude there exists $f \in I(Y_1)$ and $f \notin I(Y_2)$. Similary, there exists $g \in I(Y_2)$ and $g \notin I(Y_1)$. But then $fg \in I(Y_1) \cap I(Y_2) = I(Y)$. Hence, $I(Y)$ is not prime. $\square$

Exercises for Lecture 1: First properties of affine varieties

(1) Prove the following properties:
   (a) If $T_1 \subseteq T_2$ are subsets of $k[x_1, \dots, x_n]$, then $V(T_1) \supseteq V(T_2)$.
   (b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.
   (c) If T_1, T_2 are subsets of $k[x_1, \dots, x_n]$, then $V(T_1 \cap T_2) = V(T_1) \cup V(T_2)$.
   (d) If $T_1, T_2, \dots,$ are subsets of $k[x_1, \dots, x_n]$, then $V(\cup T_i) = \cap V(T_i)$.

(2) Decide in as many ways as you can (algebraically, geometrically) whether each of the following varieties is irreducible. For each reducible variety, determine how many irreducible components it has and what prime ideal corresponds to each component.
   (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$
   (b) $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$
   (c) $\{(x, y, z) \in \mathbb{C}^2 : x^2 + y^2 + z^2 = 0, x^2 - y^2 - z^2 + 1 = 0\}$

(3) Define X to be the zero locus of $f(x, y) = x^2 + y^2 - 1$ and $g(x) = x - 1$. Find the ideal $I(X)$. Is $I(X) = (f, g)$?