2. Outline: Dimension of affine algebraic varieties

Intuition behind the lecture: if an ant is sitting at an *average point* in Y, in *how many directions* can the ant walk?

I. What does "an average point" mean?

Definition 2.1. *A Zariski closed set is the zero locus in $\mathbb{A}^n$ of a set of polynomials $f_1, \ldots, f_k$. A Zariski open set is the complement of a Zariski closed set in $\mathbb{A}^n$.*

Remark: The Zariski open sets form a topology on $\mathbb{A}^n$ (they are closed under union and finite intersection, and contain $\emptyset$ and $\mathbb{A}^n$).

Remark: The Zariski topology on the variety $X \subseteq \mathbb{A}^n$ is the subset topology induced from the Zariski topology on $\mathbb{A}^n$. This is a language rather than a tool for geometric intuition: for instance, every nonempty Zariski open subset is dense in X!

Definition 2.2. *Property P* holds for a general point of a variety X *if there is a Zariski open subset of X on which P holds.*

Definition 2.3. *A quasi-affine variety is a Zariski open subset of an affine variety.*

II. First definition of dimension

Definition 2.4. *Let $k = \mathbb{C}$. The tangent space T_pX to the variety $X = V(f_1, \ldots, f_k)$ at the point $\mathbf{p} = (p_1, \ldots, p_n)$ consists of the points of X that simultaneously solve*

$$\left\{ \sum_j \frac{\partial f_i}{\partial x_j}(\mathbf{p})(x_j - p_j) = 0 \text{ for all } i \right\}.$$

Definition 2.5. $\dim X$ *is the minimal nonzero dimension of T_pX over all points* $\mathbf{p}$.

Definition 2.6. *The point* $\mathbf{p}$ *is a singular point of X (or a singularity) if* $\dim T_pX > \dim X$.

II. Second definition of dimension

Definition 2.7. $\dim X$ *is the dimension of a nonempty open set $U \subseteq X$ at a general point.*

III. Third definition of dimension

Definition 2.8. *The coordinate ring of the variety Y is written $k[Y]$ and is defined as*

$$k[Y] = k[x_1, \ldots, x_n]/I(Y)$$

Intuitively, the coordinate ring is the collection of distinct polynomials defined on Y. It is generated by the *coordinate functions* $x_i + I(Y)$.

Definition 2.9. *The Krull dimension of a ring A is the largest n so that A contains a chain of increasing prime ideals*

$$\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \mathfrak{p}_n \subseteq A$$

where each $\mathfrak{p}_i$ is prime and each containment is strict.

Definition 2.10. *The dimension of X is the Krull dimension of $k[X]$.*

2. Dimension of affine algebraic varieties

Dimension is a geometric notion that we all have some gut sense for—you'd balk, for instance, if I defined it in any way that made a line a two-dimensional object. Intuitively, we want dimension to answer the question: if an ant is sitting at an *average point* in Y, in *how many directions* can the ant walk? We need to make more rigorous both "an average point" and "how many direction." Over the course of doing this, we'll also make dimension more algebraic. Though this might not seem natural, we'll even give some good arguments for why the most algebraic definition is, in fact, best.

Before we begin, let's go over an example that addresses all of the topics we covered in the last lecture.

Example 2.1. *Y is the zero locus of* $\left\{ \begin{array}{l} f_1 = x^2 - yz = 0 \\ f_2 = xz - x = 0 \end{array} \right\}$

We want to do three things:

(1) *sketch the variety in $\mathbb{R}^3$,*
(2) *identify the irreducible components, and*
(3) *identify the corresponding ideals.*

Sketching the variety is a good exercise; if you do it, you should see something like the attached figure. Our guess, just looking at the picture, is that the variety has three components: the x-axis, the z-axis, and the parabola $y = x^2$ in the $z = 1$ plane. (A reasonable person could also guess that there are four components, though in our straw poll, the three-component guessers far outnumbered the four-component guessers.) The ideals of the axes are easy to determine—they're (x, z) and (x, y), respectively—and are easily identified as prime ideals. The ideal of the parabola is clearly $(z - 1, x^2 - y)$, but convincing yourself that it's prime might be harder. As a hint, we remind you of two useful facts from algebra (worth proving, if you've never done so before):

FACT: If A is a commutative ring with identity and I an ideal in A then

I is **prime** if and only if A/I is an integral domain, and
I is **maximal** if and only if A/I is a field.

We now address the question of what "an average point" could mean. To do this, we define the Zariski topology.

Definition 2.2. *A Zariski closed set is the zero locus in $\mathbb{A}^n$ of a set of polynomials $f_1, \ldots, f_k$. A Zariski open set is the complement of a Zariski closed set in $\mathbb{A}^n$.*

In the exercises from last lecture, you showed that the Zariski open sets form a topology on $\mathbb{A}^n$ (they are closed under union and finite intersection, and contain $\emptyset$ and $\mathbb{A}^n$). The Zariski topology on the variety $X \subseteq \mathbb{A}^n$ is the subspace topology induced from the Zariski topology on $\mathbb{A}^n$: Zariski closed sets in X are subvarieties of X.

There are two remarks to make. The first is completely frivolous: the Zariski topology is named for the mathematician Oscar Zariski, who worked at Johns Hopkins and then Harvard to modernize algebraic geometry (and modernize the Harvard math department). Zariski is singled out in Reid's (admittedly subjective) historical account as one of the people who rescued algebraic geometry from the French.

The second remark is a strongly-worded warning: the Zariski topology should not be used for geometric intuition. The next proposition shows that in the Zariski topology, every nonempty open subset of an irreducible variety is dense. This means that there's no such thing as a small neighborhood in the Zariski topology, and so we can't use the Zariski topology for the kinds of things we usually rely on topologies for.

Proposition 2.3. *If X is irreducible and $U \subseteq X$ is a nonempty Zariski-open set, then U is dense in X.*

Proof. Since U is nonempty, the closure $\bar{U}$ is closed and nonempty in X. The complement U^c is also closed in X, and $X = \bar{U} \cup U^c$. Since X is irreducible, we conclude $\bar{U} = X$. □

If the Zariski topology isn't used for geometric intuition, what is it good for? In short, language. The Zariski topology gives us a way to talk about 'big' and 'small' sets in a variety. The following definition, taken from Harris's *Algebraic Geometry*,

Definition 2.4. *Property P* holds for a general point of a variety *X if there is a Zariski open subset of X on which P holds.*

From the previous proposition, we know that anything which holds for the general point of an irreducible variety holds everywhere except perhaps on the boundary of a dense set. In other words, a general point of a variety means what it sounds like it should mean, with rigor leant by the Zariski topology.

In a sense, the Zariski topology (and the underlying polynomials) is the key to the rigidity of algebraic geometry. Topology has all sorts of strange counterexamples ("a space which is connected and path-connected but has an infinite dense set of singularities except on the fourth Tuesday of each month..."). This can't happen in algebraic geometry.

I'll end this topic with a definition to which I will rarely, if ever, refer. It doesn't describe a substantively new object to those we have already seen, but just gives a new name.

Definition 2.5. *A quasi-affine variety is a Zariski open subset of an affine variety.*

We'll now begin the first of three definitions of dimension. I'm going to start with the definition that is closest to definitions from outside of algebraic geometry.

Definition 2.6. *The tangent space $T_{\mathbf{p}}X$ to the variety $X = V(f_1, \ldots, f_k)$ at the point $\mathbf{p} = (p_1, \ldots, p_n)$ consists of the points of k^n that simultaneously solve*

$$\left\{ \sum_j \frac{\partial f_i}{\partial x_j}(\mathbf{p})(x_j - p_j) = 0 \text{ for all } i \right\}.$$

These are affine equations, meaning they are of the form $M\mathbf{x} + \mathbf{v} = 0$ for a coefficient matrix M and vectors $\mathbf{x}$ and $\mathbf{p}$ in k^n. Consequently, their solution set is an affine subspace of k^n, meaning a linear subspace translated by $\mathbf{p}$ away from the origin. (To belabor a point, this is an echo of the big deal I made when distinguishing between the affine variety $\mathbb{A}^n$ and the vector space k^n.) In a later lecture, we will revisit this definition and provide a more abstract version.

We make two key points about tangent spaces.

(1) There is a minimal element r of the set $\{\dim T_pX : p \in X\}$.
(2) The set $\{p \in X : \dim T_pX \text{ is minimal}\}$ is Zariski-open in X.

The first is clear, since the dimension of each T_pX is a nonnegative integer. The second involves a sneaky trick that will come up in different guises later in the semester.

Proof. **of (2):** Remember the linear algebra results that say $\dim T_pX > r$ if and only if the rank of the coefficient matrix is less than $n - r$, which in turn holds if and only if all $(n-r)\times(n-r)$ minors of the coefficient matrix have zero determinant. Since the determinant is a polynomial, this is a Zariski-closed set. □

Definition 2.7. *If X is an irreducible variety then $\dim X$ is the minimal nonzero dimension of T_pX over all points $\mathbf{p}$.*

This definition of dimension uses local rather than global data, and so varies from point to point. Hence we must specify that if X is a reducible variety, the dimension is the maximal dimension of each irreducible component.

The next definition confirms our usual usage of the words 'smooth' and 'singular'.

Definition 2.8. *The point $\mathbf{p}$ is a* **smooth point** *of X if it has $\dim T_pX = \dim X$. The point $\mathbf{p}$ is a* **singular point** *of X (or a singularity) if $\dim T_pX > \dim X$.*

Example 2.9. *Let $V(f_1, f_2)$ be the variety defined by* $\left\{ \begin{array}{l} f_1 = xz = 0, \\ f_2 = yz = 0. \end{array} \right\}$

Choose an arbitrary point $\mathbf{p} = (p_1, p_2, p_3)$ of X. The partial derivatives of f_1 and f_2 are

$$\begin{array}{lll} \frac{\partial f_1}{\partial x} = z & \frac{\partial f_1}{\partial y} = 0 & \frac{\partial f_1}{\partial z} = x \\ \frac{\partial f_2}{\partial x} = 0 & \frac{\partial f_2}{\partial y} = z & \frac{\partial f_2}{\partial z} = y \end{array}$$

and so the equations defining the tangent space to X at p are

$$\left\{ \begin{array}{l} p_3(x - p_1) + p_1(z - p_3) = 0 \\ p_3(y - p_2) + p_2(z - p_3) = 0 \end{array} \right\}.$$

If $p_1 = p_2 = 0$ *then* $\left\{ \begin{array}{c} p_3 x = 0 \\ p_3 y = 0 \end{array} \right\}$ *and the solution space is a line, unless* $p = (0,0,0)$.

By contrast, if $p_3 = 0$ *then* $\left\{ \begin{array}{c} p_1 z = 0 \\ p_2 z = 0 \end{array} \right\}$ *and the solution space is a plane, unless* $p =$ $(0,0,0)$. *Of course, if* $p = (0,0,0)$ *then the tangent space* $T_{(0,0,0)}X$ *is three-dimensional.*

Since X *is a reducible variety, we conclude that* X *has dimension two, and that* $(0,0,0)$ *is a singularity.*

Our second definition of dimension is much more handwavey, but is often used in practice.

Definition 2.10. *The irreducible variety* X *has dimension* r *if a nonempty Zariski-open subspace of* X *has dimension* r.

This is a folk definition, and is the least precise but most intuitive of the definitions we give. Its informality is analogous to some of the topological arguments identifying fundamental groups ("there are a couple cycles here and a few there, and they're *obviously* an independent generating set..."). In practice, this definition means you can disregard any exceptions or special cases when identifying the dimension of a variety from its equations.

The first example is another basic check that our definitions match our intuition.

Example 2.11. *The variety* $(\mathbb{C}^*)^r \times \mathbb{C}^s$ *has* $\mathbb{C}$ *dimension (r+s). In fact, this variety is defined by the equations* $\{x_1 \neq 0\} \cap \{x_2 \neq 0\} \cap \cdots \cap \{x_r \neq 0\}$, *which is the intersection of a finite number of Zariski-open sets in* $\mathbb{C}^{r+s}$.

Example 2.12. *Consider the variety* $V(y - x^2)$. *The Zariski-open subset* $x \neq 0$ *of the variety can be written as the union* $(\sqrt{y}, y) \cup (-\sqrt{y}, y)$. *Each part of the union is one-dimensional, so the variety is one-dimensional.*

For our third definition of dimension, we need to define coordinates on a variety. Consider the example of $V(x - y)$ in $\mathbb{A}^2$. Either x or y is a perfectly good coordinate, but they are not independent of each other because $x = y$ on the variety. A more natural choice is the coset $x + I$, which corrects the redundancy noted before. Of course, it is well-defined on the variety because every point of the variety vanishes on its ideal. The process of choosing coordinates on a general variety is essentially the same as in this example.

Definition 2.13. *The coordinate ring of the variety* X *is written* $k[X]$ *and is defined as*

$$k[X] = k[x_1, \ldots, x_n]/I(X)$$

.

Intuitively, the coordinate ring is the collection of distinct polynomials defined on X. It is generated by the *coordinate functions* $x_i + I(X)$.

Remark 2.14. *We briefly note two properties of the coordinate ring.*

(1) *The coordinate ring is the smallest ring containing all of the coordinate functions.*

(2) *Since there is a bijection between X and $I(X)$, the coordinate ring is an* invariant *of X. In particular, it does not depend on how X is embedded into $\mathbb{A}^n$.*

Example 2.15. *Two one-dimensional varieties with very different coordinate rings.*

(1) $V(y-x^2)$ *has coordinate ring* $k[x,y]/(y-x^2) \cong k[x]$.
(2) $V(xy-1)$ *has coordinate ring* $k[x,y]/(xy-1) \cong k[x,x^{-1}]$.

Our last definition of dimension is a slight modification of a purely algebraic property.

Definition 2.16. *The Krull dimension of a ring A is the largest n so that A contains a chain of increasing prime ideals*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \mathfrak{p}_n \subsetneq A$$

where each $\mathfrak{p}_i$ is prime and each containment is strict.

Note that the length of the chain counts the number of inclusions, not the number of prime ideals.

Example 2.17. *The ring $k[x_1,\ldots,x_n]$ has the following chain of prime ideals*

$$(0) \subsetneq (x_1) \subsetneq (x_1,x_2) \subsetneq \ldots, \subsetneq (x_1,\ldots,x_n) \subsetneq k[x_1,\ldots,k_n].$$

Definition 2.18. *The dimension of X is the Krull dimension of $k[X]$.*

For instance, the previous example showed that $\mathbb{A}^n$ has dimension (at most) n, which is a lot of work for a not-very-interesting result. The next proposition again is offered to demonstrate that our definition accords with what our gut tells us about dimension.

Proposition 2.19. *If $X_1 \subseteq X_2$ then* $\dim X_1 \leq \dim X_2$.

The proof will use the following, which should be proven if it is not immediately obvious to the reader.

Fact: If $r : A \mapsto B$ is any ring homomorphism and $\mathfrak{p} \leq B$ is prime then so is $r^{-1}(\mathfrak{p})$.

Proof. Let $i : X_1 \hookrightarrow X_2$ be the inclusion map. Define the map $i^* : k[X_2] \to k[X_1]$ by $i^*(f + I(X_2)) = f + I(X_1)$, so i^* is the natural map on the quotients of the polynomial ring by $I(X_2)$ and $I(X_1)$ respectively. Suppose $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq k[X_1]$ is a maximal chain of prime ideals in $k[X_1]$. Then the preimages $(i^*)^{-1}(\mathfrak{p}_0) \subsetneq \ldots \subsetneq (i^*)^{-1}(\mathfrak{p}_n) \subsetneq k[X_2]$ are a chain of prime ideals in $k[X_2]$. We conclude that $\dim X_2 \leq n$. □

This definition may seem the most distant from geometric intuition. Yet from this proposition and this third definition, we see that the dimension of a reducible variety is the maximum dimension of its components. This demonstrates, first, that the three definitions proposed here are not equivalent, if the reader had not already observed that. Second, it is a powerful indicator that this algebraic definition is better, or better at capturing global data, than the previous definitions. In the next lecture, we will continue this process of making algebraic our geometric intuition.

Lecture 2: The dimension of an affine algebraic variety

(1) Locate the singular points and sketch the following curves in $\mathbb{R}^2$.
  (a) $x^2 = x^4 + y^4$
  (b) $xy = x^6 + y^6$
  (c) $x^3 = y^2 + x^4 + y^4$
  (d) $x^2y + xy^2 = x^4 + y^4$

(2) Prove that any finite set $S \subseteq \mathbb{C}^2$ can be defined by two equations. For which fields does this hold?

(3) Let $\mathfrak{a}$ be an ideal in $k[x_1, \ldots, x_n]$ generated by r elements. Show that every irreducible component of $V(\mathfrak{a})$ has dimension at least $n - r$.